A NOTE ON UNIFORMLY CONTINUOUS SELECTIONS FOR MULTI-VALUED MAPS

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ABSTRACT. Michael's zero-dimensional selection theorem asserts that if $F\colon X \to Y$ is a lower semicontinuous multi-valued map of a zero-dimensional paracompact space X into a complete metric space Y and F has closed point values then F admits a continuous selection. If X is a zero-dimensional metric space it is known that one cannot always choose the selection to be uniformly continuous even if F is uniformly continuous with respect to the Hausdorff distance. In this note we prove that if X is an ultrametric space and F is uniformly continuous then F admits a uniformly continuous selection.

1. Introduction

Let (Y, ρ) be a metric space. For $A \subset Y$ and $\varepsilon > 0$ let $S(A, \varepsilon)$ denote the ε-ball around A. We extend the definition of Hausdorff metric as follows. For closed and non-empty subsets A, B of Y let $H_\rho(A, B)$ be the infimum of all positive δ such that $A \subset S(B, \delta)$ and $B \subset S(A, \delta)$ if such δ exists. Otherwise, let $H_\rho(A, B) = \infty$. A multi-valued map $F\colon (X, d) \to (Y, \rho)$ between metric spaces is said to be *uniformly continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $H_\rho(F(x), F(y)) < \varepsilon$.

Michael's zero-dimensional selection theorem states that if X is a paracompact zero-dimensional space and Y is a complete metric space then every multi-valued map $F\colon X \to Y$ which is lower semicontinuous and takes closed point values has a continuous selection f, i.e., a continuous function $f\colon X \to Y$ such that $f(x) \in F(x)$ for each $x \in X$.

Note that in case the space X is compact and metric the selection $f\colon X \to Y$ in Michael's theorem is obviously uniformly continuous.

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The following example shows that a selection cannot in general be chosen to be uniformly continuous even if X is a zero-dimensional metric space and F is uniformly continuous.

It is noted in [2, Example 6.1] that the multi-valued map G from the unit interval $[0,1]$ into the compact subsets of the space

$$\left\{\left(t,\sin\frac{1}{t}\right)\mid t\neq 0\right\}\cup\{(0,s)\mid -1\leq s\leq 1\}\subset\mathbb{R}^2$$

defined for every $x\in X$ by the formula

$$G(x)=\begin{cases}\{(t,\sin\frac{1}{t})\mid\frac{1}{2}x\leq t\leq x\} \text{ if } x\in(0,1],\\ \{0\}\times[-1,1] \text{ if } x=0\end{cases}$$

admits no continuous selection. Let G' be the restriction of the map G onto the set of the rational numbers from $[0,1]$. Then the map G' is uniformly continuous and is defined on the zero-dimensional space $\mathbb{Q}\cap[0,1]$ but it admits no uniformly continuous selection because otherwise it would extend to a continuous selection of G.

It is interesting to find conditions for which a multi-valued map F has a uniformly continuous selection. We give one such condition in this note.

Recall that a metric d on a space X is an *ultrametric (or non-Archimedean)* if it satisfies the strong triangle inequality

$$d(x,y)\leq\max\{d(x,z),d(z,y)\}$$

for all $x,y,z\in X$. It is known [1] that a metric space X admits an ultrametric compatible with its topology if and only if $\dim X=0$.

2. Result

Theorem 2.1. *Let (X,d) be an ultrametric space and (Y,ρ) be a complete metric space. Then every uniformly continuous multi-valued map $F\colon(X,d)\to(Y,\rho)$ with closed point values has a uniformly continuous selection.*

Proof. Let $\{\mathcal{V}_i\}_{i=1}^{\infty}$ be the family of clopen covers of the space (X,d) such that $\mathcal{V}_i$ consists of mutually disjoint balls of radius $1/i$. Let F be a uniformly continuous multi-valued map with closed point values from a metric space (X,d) to a complete metric space (Y,ρ). For every $j\in\mathbb{N}$ there is $\delta_j>0$ such that

$$H_\rho(F(x),F(y))<1/j^2$$

whenever $x,y\in X$ and $d(x,y)<\delta_j$. Let $\{\mathcal{W}_j\}_{j=1}^{\infty}$ be a subsequence of $\{\mathcal{V}_i\}_{i=1}^{\infty}$ such that $\operatorname{diam}V<\delta_j$ for every $V\in\mathcal{W}_j$. Choose $a(V,j)\in V$ for every $V\in\mathcal{W}_j$, $j\in\mathbb{N}$.

For $j=1$ let $b(V,1)\in F(a(V,1))$, $V\in\mathcal{W}_1$. Define a function $f_1\colon X\to Y$ by letting $f_1(x)=b(V,1)$ if $x\in V\in\mathcal{W}_1$. Then f_1 is well-defined (since there is a unique element of $\mathcal{W}_1$ which contains x) and uniformly continuous because $f_1(x)=f_1(y)$ whenever $d(x,y)<\delta_1$.

Suppose that for $k\in\{2,\dots,n-1\}$ there are defined uniformly continuous functions $f_k\colon X\to Y$ and points $b(V,k)\in F(a(V,k))$ such that

$$\rho(b(V,k),b(U,k-1))<1/(k-1)^2$$

and $f_k(V)=\{b(V,k)\}$ for $V\in\mathcal{W}_k$, $U\in\mathcal{W}_{k-1}$ with $V\subset U$.

For every $V\in\mathcal{W}_n$ there is a unique $U\in\mathcal{W}_{n-1}$ such that $V\subset U$. Since $\mathrm{diam}U<\delta_{n-1}$ we have $d(a(V,n),a(U,n-1))<\delta_{n-1}$ and, hence,

$$H_\rho(F(a(V,n)),F(a(U,n-1)))<1/(n-1)^2.$$

Therefore, there exists $b(V,n)\in F(a(V,n))$ such that

$$\rho(b(V,n),b(U,n-1))<1/(n-1)^2.$$

Let $f_n\colon X\to Y$ be defined by $f_n(x)=b(V,n)$ if $x\in V\in\mathcal{W}_n$. Then f_n is uniformly continuous.

By induction we obtain a Cauchy sequence $\{f_n\}$ of uniformly continuous functions which converges to some uniformly continuous function $f\colon X\to Y$. If $x\in X$ then

$$x\in\dots V_j\subset V_{j-1}\subset\dots\subset V_1$$

for some unique sequence of $V_j\in\mathcal{W}_j$, $j\in\mathbb{N}$. Therefore,

$$f(x)=\lim_{j\to\infty}f_j(x)=\lim_{j\to\infty}b(V_j,j)\in Y$$

because $\{b(V_j,j)\}$ is Cauchy and Y is complete. Since $a(V_j,j)$ converges to x and F is uniformly continuous, $F(a(V_j,j))$ converges to $F(x)$ and, hence, $f(x)=\lim_{j\to\infty}b(V_j,j)\in F(x)$. So $f(x)\in F(x)$ for every $x\in X$ and f is a uniformly continuous selection of F. □

References

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