SIMULTANEOUS BRICK PARTITIONINGS OF PEANO CONTINUA

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Abstract. Nikiel-Stasyuk-Tuncali-Tymchatyn recently proved that there exists a refining sequence of simultaneous core brick partitions of a compact graph contained in a Peano continuum. In the current note we extend this to the case of two nested Peano continua. This completes the work begun by Thomas of extending the partitioning (of a Peano continuum) results of Bing to simultaneous partitioning of two nested Peano continua. It also verifies a long-standing conjecture of Thomas.

1. Introduction

The idea of partitioning a Peano continuum goes back to Menger, Whyburn and Moise (see [3]). Bing proved that a metric space can be partitioned in [1], [2] and [3]. He proved that each Peano continuum has a sequence $\mathcal{U}_i$ of brick partitions such that $\mathcal{U}_i$ has mesh less than $1/i$ and $\mathcal{U}_{i+1}$ core refines $\mathcal{U}_i$. Among his applications of these results Bing proved that every Peano continuum admits an equivalent convex metric.

Thomas [7] proved that each pair of nested Peano continua admits a decreasing sequence of simultaneous brick partitions. In that paper he made two conjectures about nicer simultaneous partitions.

Nikiel-Stasyuk-Tuncali-Tymchatyn [6] proved that every connected, locally arc connected metric space with property S admits an equivalent convex metric thus extending Bing's convexity theorem. In the course of the proof they showed that each pair of nested Peano continua admitted a decreasing sequence of simultaneous brick partitions. This confirmed Thomas' first conjecture in [7] although that was not noticed at the time the above mentioned proof was based on Thomas' work and on the extension of Bing [3, Lemma for Theorem 8] to the case of two nested Peano continua.

On the way to proving their convexity theorem the authors in [6] proved that if N was a compact, connected graph contained in a Peano continuum M then there existed a decreasing sequence $\mathcal{U}_i$ of simultaneous brick partitions of M and N such that $\mathcal{U}_{i+1}$ was a core refinement of $\mathcal{U}_i$ (see [6, Theorem 6]). The primary purpose of this note is to show that the condition in that theorem that N be a compact graph can be relaxed to N is a Peano continuum.

We also strengthen [6, Theorem 4] to allow us to verify Thomas' second conjecture.

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2. Preliminaries

Definition 2.1. A metric space (M, d) has *property* S if for each $\varepsilon > 0$ there is a finite cover of M by connected sets of diameter less than ε.

The space $\mathbb{R}$ in its usual metric does not have property S while $(0, 1)$ in its usual metric does. So property S is a metric property. If (M, d) has property S, then it is locally connected and totally bounded. It is known that every locally connected, compact, metric space has property S.

Definition 2.2. A finite closed covering $\mathcal{V}$ of a metric space (M, d) is a *partition* of M if the following conditions are satisfied for all $U, V \in \mathcal{V}$:

- V and $\mathrm{int}(V)$ are connected and locally connected, and the first one of them is regular closed while the other is regular open,
- if $U \neq V$ then $U \cap V \subset \mathrm{bd}(U) \cap \mathrm{bd}(V)$.

If the mesh of $\mathcal{V}$ is less than $\varepsilon > 0$ then $\mathcal{V}$ is called an ε-partition of M.

Remark. Our definition of partition is different from Bing's but is equivalent to Bing's definition (see [6]).

Theorem 2.3 (Bing, [1])**.** *For a metric space M there is an ε-partitioning for each $\varepsilon > 0$ if and only if M has property S.*

Definition 2.4. We shall say that a partition $\mathcal{V}$ of M is *a brick partition* of M if the following conditions are satisfied for all $U, V \in \mathcal{V}$:

- $\mathrm{int}(V)$ is uniformly locally connected;
- $\mathrm{int}(U \cup V)$ is uniformly locally connected.

Theorem 2.5 (Bing, [1])**.** *Each Peano continuum (i.e. compact, connected and locally connected metric space) has a sequence $\{\mathcal{U}_i\}$ of brick partitionings such that $\mathcal{U}_i$ has mesh less than $1/i$ and $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$.*

Let $\mathcal{U}$ and $\mathcal{V}$ be partitions of M such that $\mathcal{V}$ refines $\mathcal{U}$. For each element $U \in \mathcal{U}$ let

$$\mathcal{V}(U) = \{V \in \mathcal{V} : V \subset U\}$$
$$B(U, \mathcal{V}) = \{V \in \mathcal{V}(U) : V \cap \mathrm{bd}(U) \neq \varnothing\}$$
$$I(U, \mathcal{V}) = \mathcal{V}(U) \setminus B(U, \mathcal{V})$$

Definition 2.6. We shall say that $\mathcal{V}$ is a *core refinement* of the partition $\mathcal{U}$ of M if $\mathcal{V}$ is a partition of M which refines $\mathcal{U}$ and the following conditions are satisfied for all $U \in \mathcal{U}$:

- $\bigcup I(U, \mathcal{V})$ is connected,
- each member of $B(U, \mathcal{V})$ meets at least one member of $I(U, \mathcal{V})$.

Definition 2.7. Let $\mathcal{U}$ be a partition of the connected, locally connected, metric space M and let N be a closed, locally connected, connected subset of M such that $\mathcal{U}|_N = \{U \cap N \mid U \in \mathcal{U}\}$ is a partition of N and for each $U \in \mathcal{U}$, $U \cap N = \mathrm{cl}(\mathrm{int}(U) \cap N)$. Then $\mathcal{U}$ is called a simultaneous partition of M and N.

Let $\mathcal{V}$ be a core refinement of $\mathcal{U}$ such that $\mathcal{V}|_N = \{V \cap N \mid V \in \mathcal{V}\}$ is a core refinement of $\mathcal{U}|_N$. Then we say that $\mathcal{V}$ is *a simultaneous core refinement* of $\mathcal{U}$ with respect to M and N.

Theorem 2.8 (Thomas, [7])**.** *Let M be a Peano continuum and N a Peano subcontinuum of M. Then there exists a sequence of simultaneous core partitionings $\mathcal{U}_i$ of M and N such that $\mathcal{U}_i$ has mesh less than $1/i$ and $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$.*

3. Main result

The following theorem is the main result of this paper.

Theorem 3.1. *Let M and N be Peano continua with $N \subset M$. Suppose that $\mathcal{U}$ is a simultaneous brick partition of M and N. Then for each $\varepsilon > 0$ there exists a simultaneous core brick partition $\mathcal{V}$ with respect to M and N such that $\mathcal{V}$ refines $\mathcal{U}$ and* $\mathrm{mesh}\mathcal{V} < \varepsilon$.

Proof. Let $\varepsilon > 0$ and let $\mathcal{U}'$ be a simultaneous brick ε-partition of M and N which refines $\mathcal{U}$ by Theorem 5 from [6].

Just suppose that there exist adjacent $U, V \in \mathcal{U}'$ with $U \cap N \neq \varnothing \neq V \cap N$ but $U \cap V \cap N = \varnothing$. Let $\mathcal{W}$ be a simultaneous brick partition of M and N which refines $\mathcal{U}'$ and such that $\mathrm{st}(U \cap V, \mathcal{W}) \cap N = \varnothing$. This is possible because $U \cap V$ and N are compact and disjoint. Let $U_N = \mathrm{st}(N \cap U, \mathcal{W}(U))$. Then U_N is connected because $N \cap U$ is connected. Let $\{C_1, \dots, C_n\}$ be the components of $U \setminus U_N$. Notice that $\mathrm{cl}(C_i) \cap N = \varnothing$ for each $i \in \{1, \dots, n\}$. Then

$$\mathcal{U}^* = (\mathcal{U}' \setminus \{U\}) \cup \{U_N, \mathrm{cl}(C_1), \dots, \mathrm{cl}(C_n)\}$$

is a simultaneous brick ε-partition of M and N which refines $\mathcal{U}$ and which has at least one fewer pairs of adjacent elements each of which meets N but which contains no point of N in their intersection. By the finiteness of $\mathcal{U}'$ we may suppose that if U and V are adjacent elements of $\mathcal{U}'$ each of which meets N then $U \cap V \cap N \neq \varnothing$.

For each $U \in \mathcal{U}$ let $T_{U,N}$ be a compact connected graph in $\mathrm{int}(U) \cap N$ such that $T_{U,N} \cap \mathrm{int}(V)$ is connected and non-empty for each $V \in \mathcal{U}'(U)$ whenever $N \cap V \neq \varnothing$. Also $V, W \in \mathcal{U}'(U)$ are adjacent and $V \cap N \neq \varnothing \neq W \cap N$ then

$$(\dagger) \qquad \varnothing \neq T_{U,N} \cap V \cap W \subset \mathrm{int}_N((V \cup W) \cap N).$$

We require $T_U \cap V \cap N$ connected for each $V \in \mathcal{U}'(U)$ and $T_U \cap N$ is connected.

For each $U \in \mathcal{U}$ let T_U be a compact connected graph in $\mathrm{int}(U)$ such that $T_{U,N} \subset T_U$, $T_U \cap \mathrm{int}(\mathrm{V})$ is non-empty and connected and $T_U \cap V$ is connected for each $V \in \mathcal{U}'(U)$. If $V, W \in \mathcal{U}'(U)$ are adjacent and $T_U \cap V \cap W \neq \varnothing$ then $\varnothing \neq T_U \cap V \cap W \subset \mathrm{int}(V \cup W)$.

Let

$$0 < \delta < \frac{1}{3} \min\{d(T_U \cup \bigcup I(U, \mathcal{U}'), \mathrm{bd}(U)) \mid U \in \mathcal{U}\}.$$

Let $\mathcal{U}''$ be a simultaneous brick δ-partition of M and N which refines $\mathcal{U}'$ and is such that

$$(\dagger\dagger) \qquad \text{if } V, W \in \mathcal{U}'' \text{ are adjacent and each meets } N \text{ then so does } V \cap W.$$

For each $U \in \mathcal{U}$ and $V \in B(U, \mathcal{U}')$ let

$$U_c = \mathrm{st}\left(T_U \cup \bigcup I(U, \mathcal{U}'), \mathcal{U}''\right),$$
$$S_V = \mathrm{st}(T_U \cap V, \mathcal{U}''(V))$$

and

$$V_c = S_V \cup \{K \mid K \text{ component of } V \setminus S_V,\ K \cap \mathrm{bd}(U) = \varnothing\}.$$

Note that U_c, $U_c \cap N$, V_c and $V_c \cap N$ are connected for each $V \in \mathcal{U}'(U)$, $U \in \mathcal{U}$ because $\mathcal{U}, \mathcal{U}'$ and $\mathcal{U}''$ are simultaneous brick partitions, $\mathcal{U}'$ satisfies ($\dagger$), $\mathcal{U}''$ satisfies ($\dagger\dagger$) and by construction.

If K is a component of $V \setminus S_V$ then $\mathrm{cl}(K) \cap S_V \neq \varnothing$ because V is connected and S_V is closed. Hence, V_c is connected.

If K is a component of $V \setminus S_V$ which meets N then either $N \cap K = N \cap V \subset K$ as $N \cap V$ is connected or $\mathrm{cl}(K \cap N) \cap S_V \neq \varnothing$. In the latter case for each component L of $N \cap K$, $\mathrm{cl}(L) \cap S_V \neq \varnothing$ by the boundary bumping theorem because $N \cap V$ is connected and meets S_V. So $N \cap V_c$ is connected.

Suppose now that there exists $U \in \mathcal{U}$ and $V \in B(U,\mathcal{U}')$ and a component K of $(N \cap V) \setminus V_c$ such that $K \cap \mathrm{bd}(U) = \varnothing$. Let

$$0 < \delta' < \frac{1}{3} \min \{d(K, \mathrm{bd}(U)) \mid K \text{ component of } (V \cap N) \setminus V_c,\, K \cap \mathrm{bd}(U) = \varnothing,\\ V \in B(U,\mathcal{U}'),\, U \in \mathcal{U}\} \cup \{d(K,L) \mid K,\, L \text{ component of } (V \cap N) \setminus V_c,\\ \mathrm{cl}(K) \cap \mathrm{cl}(L) = \varnothing,\, V \in B(U,\mathcal{U}'),\, U \in \mathcal{U}\}\,.$$

Let $\mathcal{U}'''$ be a simultaneous brick δ'-partition of M and N that refines $\mathcal{U}''$. If $V \in B(U,\mathcal{U}')$ with $U \in \mathcal{U}$ let

$$V_c' = V_c \cup \bigcup \{\mathrm{st}(\mathrm{cl}(K), \mathcal{U}'''(V)) \mid K \text{ component of } (V \cap N) \setminus V_c,\, K \cap \mathrm{bd}(U) = \varnothing\}.$$

Note that V_c' is closed and connected and $V_c' \cap N$ is connected by the same reasoning as for $V_c \cap N$. Let

$$V_c'' = V_c' \cup \bigcup \{W \mid W \text{ component of } V \setminus V_c',\, W \cap \mathrm{bd}(U) = \varnothing\}.$$

If $V \cap N \neq \varnothing$ then $V_c'' \cap N = V_c' \cap N$ because $V \cap N$ is connected and $V_c' \cap N \neq \varnothing$ and by a similar argument to the one for $V_c \cap N$. Also, V_c'' is connected. We set $V_c'' \in I(U, \mathcal{V})$ where $\mathcal{V}$ is the desired partition.

Let $\{K_{V,1}, \ldots, K_{V,n_V}\}$ be the components of $V \setminus V_c''$ such that $K_{V,i} \cap N$ meets $\mathrm{bd}(U)$. Note that $\mathrm{cl}(K_{V,i} \cap N) \cap (V_c'' \cap N) \neq \varnothing$. Let $L_{V,1}, \ldots, L_{V,m_V}$ be the components of

$$V \setminus (V_c'' \cup \mathrm{cl}(K_{V,1}) \cup \cdots \cup \mathrm{cl}(K_{V,n_V})).$$

Then $\mathrm{cl}(L_{V,i})$ meets V_c'' since V is connected. Also note that $L_{V,i} \cap \mathrm{bd}(U) \neq \varnothing$ and $L_{V,i} \cap N = \varnothing$. We set $\mathrm{cl}(L_{V,i}) \in B(U, \mathcal{V})$.

Generally we define the brick partition $\mathcal{V}$ refining $\mathcal{U}$ as follows:

$$\mathcal{V} = \bigcup \{I(U,\mathcal{U}') \mid U \in \mathcal{U}\} \cup \bigcup \{V \in B(U,\mathcal{U}') \mid V \cap N = \varnothing,\, U \in \mathcal{U}\} \cup \\ \bigcup \{\mathrm{cl}(K_{V,i}) \mid K_{V,i} \subset V \in B(U,\mathcal{U}'), V \cap N \neq \varnothing,\, U \in \mathcal{U}\} \cup \\ \bigcup \{\mathrm{cl}(L_{V,i}) \mid L_{V,i} \subset V,\, V \in B(U,\mathcal{U}')),\, U \in \mathcal{U}\} \cup \\ \bigcup \{V_c'' \mid V \in B(U,\mathcal{U}'),\, U \in \mathcal{U}\}.$$

From the construction we see that if $V \in I(U,\mathcal{U}')$, $U \in \mathcal{U}$ then $V \in I(U, \mathcal{V})$. Therefore $\mathcal{V}$ is the required simultaneous core brick ε-partition of M and N which refines $\mathcal{U}$. □

Theorem 3.2. *Let $N \subset M$ be Peano continua and let H and K be disjoint closed subsets of M. Then there is a continuous function $T \colon M \to [0,1]$ such that*

1) $T(H) = 0$ *and* $T(K) = 1$,
2) *for each connected and non-degenerate set $W \subset [0,1]$, the sets*

$$T^{-1}(W) \text{ and } T^{-1}(W) \cap N$$

have property S,

3) *for all but at most countably many* $x \in [0,1]$ *the sets*

$$T^{-1}([0,x)),\ T^{-1}([0,x)) \cap N,\ T^{-1}((x,1]) \text{ and } T^{-1}((x,1]) \cap N$$

are uniformly locally connected,

4) *for all but at most countably many* $x \in [0,1]$ *the sets*

$$\mathrm{cl}(T^{-1}([0,x)) \cap N) = T^{-1}([0,x]) \cap N$$

and

$$T^{-1}([x,1]) \cap N = \mathrm{cl}(T^{-1}((x,1]) \cap N).$$

The proof is essentially the same as before. This verifies Thomas's second conjecture.

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