

A NON-CHAINABLE PLANE CONTINUUM WITH SPAN ZERO

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ABSTRACT. A plane continuum is constructed which has span zero but is not chainable.

1. INTRODUCTION

1.1. **Background.** The notion of the span of a continuum was introduced by Lelek in [8]. There he proved that chainable continua have span zero, and in 1971 ([9]) he asked whether the converse also holds. This is known as Lelek's problem, and has become a topic of much interest in continuum theory, in part because there are few other means presently available to decide whether a given continuum is chainable. An affirmative answer to Lelek's problem would have provided a useful tool with applications to other open problems in continuum theory; for instance, it would have completed the classification of planar homogeneous continua (see [20]).

Lelek's problem has been featured in a number of recent surveys, appearing as Problem 8 in [4], Problem 2 in [7], Problem 81 in [5], Conjecture 2 in [12], and in [15, p. 255].

There has been previous work toward finding a counterexample for Lelek's problem. Repovš et al. exhibit in [21] a sequence of trees in the plane with arbitrarily small (positive) spans, none of which has a chain cover of mesh < 1 . In [1], Bartošová et al. consider generalizations of the notions of chainability and span zero to the class of Hausdorff (not necessarily metrizable) continua, and prove via a model-theoretic construction that a counterexample for Lelek's problem in that context would imply that there exists a metric counterexample.

Many positive partial results for Lelek's problem have been obtained in [13], [16], [17], and [20]. Notably, Minc proves in [13] that span zero is equivalent to chainability among those continua which are inverse limits of trees with simplicial bonding maps, and Oversteegen does the same in [16] for continua which are the image of a chainable continuum under an induced map.

A number of properties of chainable continua have been established for span zero continua. It is known that span zero continua are atriodic [8], and Oversteegen and Tymchatyn show in [19] that they are tree-like and weakly chainable. Further, Marsh proves in [11] that products of span zero continua have the fixed point property, and Bustamante et al. prove in [3] theorems about fixed point and universality properties in the hyperspace of subcontinua of a span zero continuum, generalizing corresponding theorems for chainable continua.

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In this paper, we give an example showing that in general span zero does not imply chainable, even among continua in the plane. This example also provides a negative answer to a question of Mohler (Problem 171 of [5], Problem 7 of [10], and Problem 32 of [22]), which asks whether every weakly chainable atriodic tree-like continuum is chainable.

The example presented here is a simple-triod-like continuum, which we will develop as a nested intersection of thickened simple triods in the plane. In Section 2, we introduce some terminology that is useful for describing these simple triods in a combinatorial way. We then show how to extract combinatorial information from a given chain cover of a graph described this way in Section 3 (see [16] for some related work). Section 4 contains the necessary combinatorial lemmas pertaining to our particular graphs, and in Section 5 we construct the example precisely and prove it has the stated properties.

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1.2. Definitions and notation. A *continuum* is a compact connected metric space. We will always denote the metric by d .

Given a continuum X , the *span of X* is the supremum of all $\eta \geq 0$ for which there exists a subcontinuum Z of $X \times X$ such that: 1) $d(x, y) \geq \eta$ for each $(x, y) \in Z$; and 2) $\pi_1(Z) = \pi_2(Z)$, where $\pi_1, \pi_2 : X \times X \rightarrow X$ are the first and second coordinate projections, respectively.

The following properties are straightforward (see [8]):

- if X and Y are continua with $X \subseteq Y$, then $\text{span}(X) \leq \text{span}(Y)$;
- the arc $[0, 1]$ has span zero; and
- if $\langle X_n \rangle_{n=1}^\infty$ is a sequence of continua in a given compact metric space, then

$$\limsup_{n \rightarrow \infty} \text{span}(X_n) \leq \text{span}(\limsup_{n \rightarrow \infty} X_n).$$

The third property implies in particular that given any continuum $X \subset \mathbb{R}^2$ and any $\varepsilon > 0$, there is some $\delta > 0$ such that $\text{span}(\overline{X}_\delta) < \text{span}(X) + \varepsilon$, where \overline{X}_δ denotes the δ -neighborhood of X in \mathbb{R}^2 .

A *chain cover* of a continuum X is a finite open cover $\mathcal{U} = \langle U_\ell : 0 \leq \ell < L \rangle$ which is enumerated in such a way that $U_{\ell_1} \cap U_{\ell_2} \neq \emptyset$ if and only if $|\ell_1 - \ell_2| \leq 1$. X is *chainable* if every open cover of X has a refinement which is a chain cover.

A *simple triod* is a continuum T which is the union of three arcs, A_1, A_2, A_3 , which have a common endpoint o and are otherwise pairwise disjoint. A_1, A_2, A_3 are called the *legs* of T , and o is the *branch point* of T .

If $f : X \rightarrow Y$ is a function and $x_1, \dots, x_n \in X$, we will often write

$$x_1 \cdots x_n \xrightarrow{f} y_1 \cdots y_n$$

to mean $f(x_i) = y_i$ for each i .

Given a set S , a *total quasi-order on S* is a binary relation \leq on S which is reflexive and transitive, and which satisfies the property that for every $s_1, s_2 \in S$, we have $s_1 \leq s_2$ or $s_2 \leq s_1$ (or both). If \leq is a total quasi-order, we write $s_1 \simeq s_2$ to mean $s_1 \leq s_2$ and $s_2 \leq s_1$, and we write $s_1 < s_2$ to mean $s_1 \leq s_2$ and $s_2 \not\leq s_1$. Elements s_1 and s_2 are *\leq -adjacent* if $s_1 \not\leq s_2$ and there is no $s \in S$ with $s_1 < s < s_2$ or $s_2 < s < s_1$.

If S is finite and \leq is a total quasi-order on S , then there is a function $f : S \rightarrow \mathbb{Z}$ which is order preserving (i.e. $f(s_1) \leq f(s_2)$ if and only if $s_1 \leq s_2$) whose range is a contiguous block of integers.

By a *graph*, we will mean an undirected connected graph without multiple edges joining the same pair of vertices, and without any edge from a vertex to itself. If G is a graph, $V(G)$ denotes the set of vertices. A pair of vertices $v_1, v_2 \in V(G)$ is *adjacent in G* provided there is an edge between them. A sequence of distinct vertices $v_1, \dots, v_n \in V(G)$ is *consecutive in G* provided there is an edge between v_i and v_{i+1} for each $0 \leq i \leq n-1$.

A graph G will be considered as a topological space in the usual way, where the edges are realized by arcs. If $v_1, v_2 \in V(G)$ are adjacent in G , then we will use the notation $[v_1, v_2]_G$ to denote the arc joining v_1 and v_2 ; we will often drop the subscript G when the graph is clear from the context.

A *tree* is a continuum-theoretic graph (i.e. a connected finite union of arcs with pairwise finite intersections) which contains no circle. If T is a tree and $a, b \in T$, then $[a, b]_T$ denotes the minimal arc $A \subseteq T$ with $a, b \in A$; again, we will often drop the subscript T when the tree is clear from the context.

By a *word*, we will mean a finite sequence of symbols from some alphabet Γ , where an *alphabet* is any set of symbols not including $(,), [,], \prod, \cap$, and \leftarrow . If ω is a word, then $|\omega|$ denotes the length of ω . A word ω will be considered as a function on the set of integers $\{0, 1, \dots, |\omega| - 1\}$. ω^\leftarrow denotes the reverse of ω , defined by $\omega^\leftarrow(j) = \omega(|\omega| - j - 1)$.

If $\omega_1, \dots, \omega_n$ is a sequence of words, then $\prod_{i=1}^n \omega_i$ denotes the concatenation of these words. If ω is a word and n is a non-negative integer, then ω^n denotes the word obtained by repeating ω n times; that is, $\omega^n = \prod_{i=1}^n \omega$. Given words ω_1, ω_2 such that the last symbol of ω_1 coincides with the first symbol of ω_2 , define $\omega_1 \cap \omega_2$ to be the word obtained by concatenating onto ω_1 all but the first symbol in ω_2 . For example, $abc \cap caba = abcaba$.

2. GRAPH-WORDS

2.1. Sketches and the graph-word ρ_N .

Definition. A *graph-word in the alphabet Γ* is a pair $\rho = \langle G_\rho, w_\rho \rangle$ where G_ρ is a graph, and $w_\rho : V(G_\rho) \rightarrow \Gamma$ is a function.

Let us fix, for the rest of this paper, the alphabet $\Gamma := \{a, b, c\} \cup \{b_t : t \in [0, 1]\}$.

For each positive integer N , denote by $\alpha_N, \beta_N, \gamma_N$ the following three words:

$$\begin{aligned} & (abc)^{2N+1} \prod_{i=0}^{2N-1} [ab_{i/2N}cb_{i/2N}a(cba)^{2N-i-1}cbc(abc)^{2N-i-1}] ab_1cb_1a(cba)^{2N+1} \\ & (abc)^{2N+1} \prod_{i=0}^{2N-1} [ab_{i/2N}cb_{i/2N}a(cba)^{2N-i-1}cbabc(abc)^{2N-i-1}] ab_1cb_1a(cba)^{2N+1}cb \\ & ac \end{aligned}$$

For later use, we also define the word β_N^- to be identical to the word β_N except without the final b .

Define the graph-word ρ_N as follows. Let G_{ρ_N} be a simple triod, with vertex set $V(G_{\rho_N}) = \{o, p_1, \dots, p_{|\alpha_N|-1}, q_1, \dots, q_{|\beta_N|-1}, r\}$, where o is the branch point of the triod, $p_{|\alpha_N|-1}, q_{|\beta_N|-1}, r$ are the endpoints of G_{ρ_N} , the points p_j belong to the

leg $[o, p_{|\alpha_N|-1}]$ with $p_j \in [o, p_{j+1}]$ for each j , and the points q_j belong to the leg $[o, q_{|\beta_N|-1}]$ with $q_j \in [o, q_{j+1}]$ for each j . Put $p_0 := o$ and $q_0 := o$. Define $w_{\rho_N} : V(G_{\rho_N}) \rightarrow \Gamma$ by $w_{\rho_N}(p_j) := \alpha_N(j)$, $w_{\rho_N}(q_j) := \beta_N(j)$, and $w_{\rho_N}(r) := \gamma_N(1) = c$.

To construct the example of a non-chainable continuum X with span zero, we will define a sequence of simple triods $\langle T_N \rangle_{N=0}^\infty$ in the plane such that T_N is contained in a small neighborhood of T_{N-1} in \mathbb{R}^2 for each $N > 0$; X will then be defined as the intersection of the nested sequence of neighborhoods of the triods T_N . The graph-word ρ_N will be used to describe the pattern with which we nest the simple triod T_N inside a small neighborhood of T_{N-1} . To carry this out precisely, we introduce the notion of a *sketch* below.

Remark. The space X may alternatively be described as an inverse limit of simple triods, as follows. Let T be a simple triod with endpoints denoted as a, b, c and branch point o . Denote a point in the interior of the arc $[o, b]$ by b_0 , and parameterize the arc $[b_0, b]$ by b_t for $t \in [0, 1]$, so that $b_1 = b$ (as per the notion of a Γ -marking defined below). Then the N -th bonding map $b_N : T \rightarrow T$ takes o to a , is the identity on the segment $[b_0, b]$, and otherwise maps the legs $[o, a]$, $[o, b]$, $[o, c]$ in a piecewise linear way according to the patterns $\alpha_N, \beta_N, \gamma_N$, respectively. Figures 1, 2, and 3, along with the proof of Proposition 1 below, provide some geometric intuition for how this looks.

Definition. Given a simple triod T with branch point o , a Γ -marking of T is a function $\iota : \Gamma \rightarrow T$ such that $\iota(a), \iota(b), \iota(c)$ are the endpoints of T and $\{\iota(b_t) : t \in [0, 1]\} \subset [o, \iota(b)]$ are such that whenever $t < t'$, we have $\iota(b_t) \in [o, \iota(b_{t'})]$ and $\text{diam}([\iota(b_t), \iota(b_{t'})]) = d(\iota(b_t), \iota(b_{t'})) = t' - t$.

Define the simple triod $T_0 := \{(x, 0) : x \in [-1, 1]\} \cup \{(0, y) : y \in [0, 2]\} \subset \mathbb{R}^2$, and define a Γ -marking $\iota : \Gamma \rightarrow T_0$ by:

$$\begin{aligned} \iota(a) &:= (-1, 0) \\ \iota(b) &:= (0, 2) \\ \iota(c) &:= (1, 0) \\ \iota(b_t) &:= (0, 1 + t) \text{ for } t \in [0, 1]. \end{aligned}$$

Definition. Define the equivalence relation \approx_Γ on Γ by $\sigma \approx_\Gamma \tau$ if and only if $\sigma = \tau$ or $\sigma, \tau \in \{b\} \cup \{b_t : t \in [0, 1]\}$.

The relation \approx_Γ partitions Γ into three equivalence classes. If ι is a Γ -marking of a triod T , then $\sigma \approx_\Gamma \tau$ if and only if $\iota(\sigma)$ and $\iota(\tau)$ belong to the same leg of T .

To simplify definitions and arguments in the following, we will restrict our attention to a special class of graph-words.

Definition. A *compliant graph-word* is a graph-word $\langle G, w \rangle$ in the alphabet Γ such that there is no pair of adjacent vertices v_1, v_2 in G with $w(v_1) \approx_\Gamma w(v_2)$.

Observe that ρ_N is a compliant graph-word for each N .

Definition. Suppose T is a simple triod with a Γ -marking $\iota : \Gamma \rightarrow T$, and let $\rho = \langle G, w \rangle$ be a compliant graph-word in the alphabet Γ . Then $\widehat{w} : G \rightarrow T$ is a ρ -suggested bonding map provided $\widehat{w}|_{V(G)} = \iota \circ w$, and for any adjacent $v_1, v_2 \in V(G)$, we have that $\widehat{w}|_{[v_1, v_2]_G}$ is a homeomorphism from $[v_1, v_2]_G$ to $[\iota(w(v_1)), \iota(w(v_2))]_T$.

Definition. Let $\langle \Omega, d \rangle$ be a metric space, let $T \subseteq \Omega$ be a Γ -marked simple triod, let $G \subseteq \Omega$ be a graph, and let $\varepsilon > 0$. Then $\rho = \langle G, w \rangle$ is a $\langle T, \varepsilon \rangle$ -sketch of G in Ω if ρ is a compliant graph-word in the alphabet Γ , and there is a ρ -suggested bonding map $\widehat{w} : G \rightarrow T$ such that $d(x, \widehat{w}(x)) < \frac{\varepsilon}{2}$ for every $x \in G$.

The next proposition assures us that we may use the graph word ρ_N defined above to describe the pattern with which we embed one simple triod into a small neighborhood of another, in the plane.

We will need some additional notation when working with the graph-word ρ_N . For each $i \leq 2N$, define $n(i)$ and $m(i)$ to be the unique integers such that

$$\begin{aligned} (n(i) - 1) n(i) (n(i) + 1) &\stackrel{\alpha_N}{\mapsto} b_{i/2N} c b_{i/2N} \\ (m(i) - 1) m(i) (m(i) + 1) &\stackrel{\beta_N}{\mapsto} b_{i/2N} c b_{i/2N}. \end{aligned}$$

For each $i < 2N$, define $\theta(i) := 6N - 3i + 1$ and $\phi(i) := 6N - 3i + 2$. Observe that

$$\begin{aligned} (n(i) + \theta(i) - 1) (n(i) + \theta(i)) (n(i) + \theta(i) + 1) &\stackrel{\alpha_N}{\mapsto} cbc \\ (m(i) + \phi(i) - 2) (m(i) + \phi(i) - 1) \cdots (m(i) + \phi(i) + 2) &\stackrel{\beta_N}{\mapsto} cbabc. \end{aligned}$$

Note that $n(0) = m(0) = 6N + 5$, and that $n(i) + 2\theta(i) = n(i + 1)$ and $m(i) + 2\phi(i) = m(i + 1)$ for each $i < 2N$.

When discussing an embedding of the graph G_{ρ_N} in \mathbb{R}^2 , we will make no notational distinction between the points of G_{ρ_N} before and after the embedding.

Proposition 1. *Suppose $T \subset \mathbb{R}^2$ is a simple triod and $\iota : \Gamma \rightarrow T$ is a Γ -marking. For any integer $N > 0$ and any $\varepsilon > 0$, there is an embedding of the simple triod graph G_{ρ_N} in \mathbb{R}^2 such that ρ_N is a $\langle T, \varepsilon \rangle$ -sketch of G_{ρ_N} in \mathbb{R}^2 . Moreover, the embedding can be chosen such that $q_{|\beta_N|-1} = \iota(b)$, $q_{|\beta_N|-2} = \iota(c)$, and $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{G_{\rho_N}} = [\iota(c), \iota(b)]_T$.*

Proof. For simplicity, we will argue only the case $T = T_0$, with the Γ -marking ι as described above; the general case can be treated similarly.

First we will analytically define a different embedding of G_{ρ_N} in \mathbb{R}^2 , then we will describe how to obtain the desired embedding from it.

Let $\eta > 0$ be significantly smaller than ε , say $\eta < \frac{\varepsilon}{20N^2}$. For $0 \leq i \leq 2N$, put

$$\begin{aligned} p_{n(i)} &:= (1 + \eta, (4i + \frac{3}{2})\eta), \\ q_{m(i)} &:= (1, (4i + \frac{3}{2})\eta). \end{aligned}$$

For $0 \leq i < 2N$ and $1 \leq j < \theta(i)$, put

$$\begin{aligned} p_{n(i)+j} &:= (1 - j, (4i + 3)\eta), \\ p_{n(i+1)-j} &:= (1 - j, 4(i + 1)\eta), \end{aligned}$$

and put $p_{n(i)+\theta(i)} := (1 - \theta(i), (4i + \frac{7}{2})\eta)$. For $0 \leq i < 2N$ and $1 \leq j < \phi(i)$, put

$$\begin{aligned} q_{m(i)+j} &:= (1 - j, (4i + 2)\eta), \\ q_{m(i+1)-j} &:= (1 - j, (4(i + 1) + 1)\eta), \end{aligned}$$

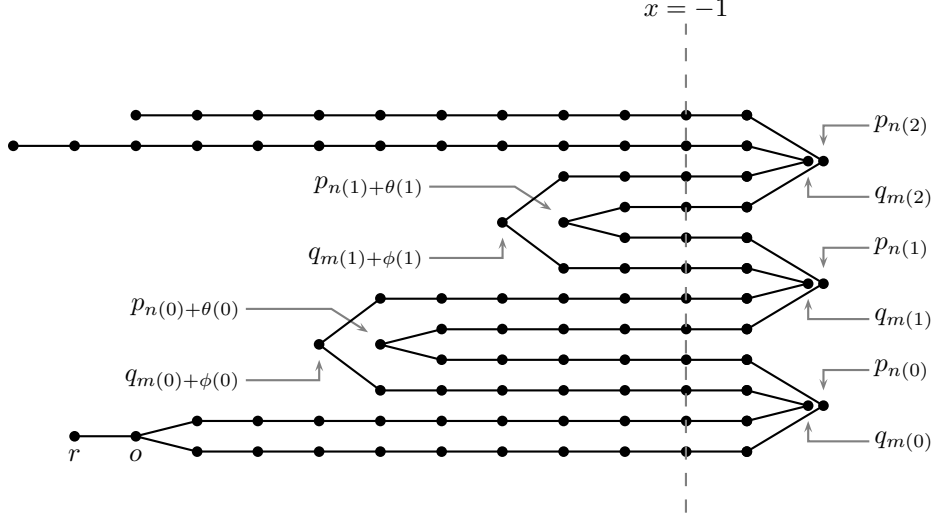


FIGURE 1. The intermediate stage G' for the embedding of G_{ρ_1} in \mathbb{R}^2 .

and put $q_{m(i)+\phi(i)} := (1 - \phi(i), (4i + \frac{7}{2})\eta)$. Further, put

$$\begin{aligned} p_{n(0)-j} &:= (1 - j, 0) && \text{for } 1 \leq j < 6N + 5, \\ q_{m(0)-j} &:= (1 - j, \eta) && \text{for } 1 \leq j < 6N + 5, \\ q_{m(2N)+j} &:= (1 - j, (8N + 2)\eta) && \text{for } 1 \leq j \leq 6N + 7, \\ p_{n(2N)+j} &:= (1 - j, (8N + 3)\eta) && \text{for } 1 \leq j \leq 6N + 5. \end{aligned}$$

Finally, put $o := (-6N - 4, \frac{1}{2}\eta)$ and $r := (-6N - 5, \frac{1}{2}\eta)$. Join each pair of adjacent vertices in G_{ρ_N} by a straight line segment in \mathbb{R}^2 . Denote the resultant embedding of G_{ρ_N} in \mathbb{R}^2 by G' . Figure 1 depicts the embedding G' for $N = 1$.

Observe that in G' , for each integer $k \leq -1$, if v and v' are two vertices in the line $x = k$, then $w(v) = w(v')$. Also notice that each vertex v in the line $x = -1$ is already close to the point $\iota(w(v)) = \iota(a) = (-1, 0)$, and that each vertex u of the form $p_{n(i)}$ or $q_{m(i)}$ is already close to the point $\iota(w(u)) = \iota(c) = (1, 0)$. We now describe heuristically in two steps how to mold G' into the embedding we seek.

First, for each $i \leq 2N$, for each triple $\langle v_1, v_2, v_3 \rangle$ of the form $\langle p_{n(i)-2}, p_{n(i)-1}, p_{n(i)} \rangle$, $\langle q_{m(i)-2}, q_{m(i)-1}, q_{m(i)} \rangle$, $\langle p_{n(i)+2}, p_{n(i)+1}, p_{n(i)} \rangle$, or $\langle q_{m(i)+2}, q_{m(i)+1}, q_{m(i)} \rangle$, move the vertex v_2 up to be close to the point $\iota(b_{i/2N})$, move the vertex v_3 down slightly, and shape the arcs joining v_1 to v_2 and v_2 to v_3 so that:

- (1) there is a homeomorphism $\hat{w}_1 : [v_1, v_2]_{G'} \rightarrow [\iota(a), \iota(b_{i/2N})]_{T_0}$ such that $\hat{w}_1(v_1) = \iota(a)$, $\hat{w}_1(v_2) = \iota(b_{i/2N})$, and $d(x, \hat{w}_1(x)) < \eta$ for each $x \in [v_1, v_2]_{G'}$,
- (2) there is a homeomorphism $\hat{w}_2 : [v_2, v_3]_{G'} \rightarrow [\iota(b_{i/2N}), \iota(c)]_{T_0}$ such that $\hat{w}_2(v_2) = \iota(b_{i/2N})$, $\hat{w}_2(v_3) = \iota(c)$, and $d(x, \hat{w}_2(x)) < \eta$ for each $x \in [v_2, v_3]_{G'}$, and
- (3) $[v_1, v_2]_{G'} \cup [v_2, v_3]_{G'}$ misses the closed upper-right quadrant of the plane $\{(x, y) : x \geq 0, y \geq 0\}$,

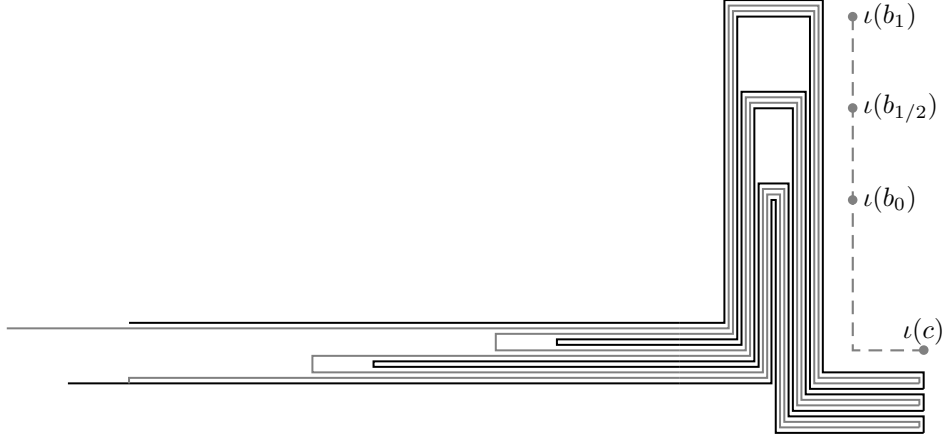


FIGURE 2. The second intermediate stage for the embedding of G_{ρ_1} in \mathbb{R}^2 .

and so that in the end no new intersections between those arcs have been introduced (i.e., so that the result is still an embedding of G_{ρ_N}). Figure 2 depicts the result for $N = 1$.

Next, take the strip $\{(x, y) : x \leq -1, 0 \leq y \leq (8N + 3)\eta\}$ and stretch and wind it counter-clockwise $2N + 2$ times around the outside of

$$\bigcup_{i=0}^{2N} ([p_{n(i)-2}, p_{n(i)+2}]_{G'} \cup [q_{m(i)-2}, q_{m(i)+2}]_{G'}),$$

so that for each integer $k \leq -1$, all the vertices v in the line $x = k$ end up near the point $\iota(w(v)) \in T_0$, taking care to make sure $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{G'} = [\iota(c), \iota(b)]_{T_0}$. Figure 3 depicts roughly how this wrapping looks.

The resulting embedding satisfies the desired properties. \square

2.2. Span and ρ_N . In this section we prove that the span of a simple triod embedded in a way described by ρ_N converges to 0 as $N \rightarrow \infty$. This will ensure that we will obtain a continuum with span zero when we take the nested intersection of neighborhoods of triods embedded in \mathbb{R}^2 as described by the ρ_N 's.

Given a set T and a subset $S \subseteq T \times T$, let $S^{-1} = \{(x, y) \in T \times T : (y, x) \in S\}$.

Lemma 2. *Let T be a simple triod with legs A_1, A_2, A_3 and branch point o . For each $i \in \{1, 2, 3\}$ let p_i be the endpoint of leg A_i other than o . Suppose $\delta > 0$ and $W \subset A_1 \times A_2$ is an arc such that $(o, o) \in W$, W meets $(\{p_1\} \times A_2) \cup (A_1 \times \{p_2\})$, and $d(x_1, x_2) \leq \delta$ for each $(x_1, x_2) \in W$. Then the span of T is $\leq \delta$.*

Proof. Suppose $Z \subset T \times T$ is a subcontinuum with $\pi_1(Z) = \pi_2(Z)$. If $\pi_1(Z)$ is an arc, then it is easy to see that Z meets the diagonal $\Delta T = \{(x, x) : x \in T\}$, as arcs have span zero.

If $\pi_1(Z)$ is a subtrioid T' of T , then we may assume $T = T'$ by replacing the arc W by the component of $W \cap (T' \times T')$ that contains (o, o) . Let K_1 and K_2 be disjoint closed and open subsets of $(A_1 \times A_2) \setminus W$ such that $(A_1 \times \{o\}) \setminus W \subset K_1$, $(\{o\} \times A_2) \setminus W \subset K_2$, and $K_1 \cup K_2 = (A_1 \times A_2) \setminus W$.

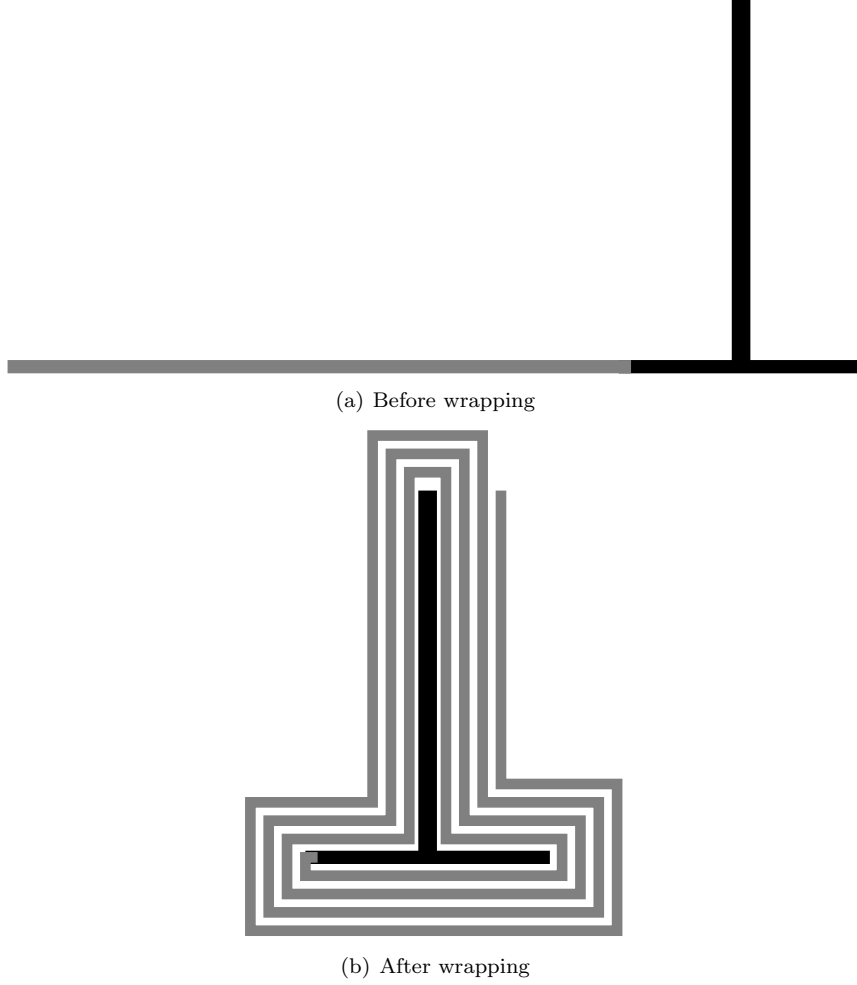


FIGURE 3. Wrapping the strip counterclockwise around the simple triod to obtain the embedding of G_{ρ_N} in \mathbb{R}^2 .

For each $i \in \{1, 2, 3\}$ let U_i and V_i be the two components of $(A_i \times A_i) \setminus \Delta T$, where $(A_i \setminus \{o\}) \times \{o\} \subset U_i$ and $\{o\} \times (A_i \setminus \{o\}) \subset V_i$. It can then be seen that the set

$$Y := (U_1 \cup U_2 \cup V_3 \cup (A_1 \times A_3) \cup (A_2 \times A_3) \cup K_1 \cup K_2^{-1}) \setminus W$$

is closed and open in $(T \times T) \setminus (W \cup W^{-1} \cup \Delta T)$ (see Proposition 5.1 of [6]). Note that $(T \times T) \setminus (W \cup W^{-1} \cup \Delta T) = Y \cup Y^{-1}$.

Observe that $p_3 \notin \pi_1(Y)$ and $p_3 \notin \pi_2(Y^{-1})$, hence $Z \not\subset Y$ and $Z \not\subset Y^{-1}$. Since Z is connected, it follows that Z must meet $W \cup W^{-1} \cup \Delta T$.

Thus in either case, there is some $(x_1, x_2) \in Z$ with $d(x_1, x_2) \leq \delta$. Therefore T has span $\leq \delta$. \square

Proposition 3. *Suppose $T \subset \mathbb{R}^2$ is Γ -marked. If the triod graph G_{ρ_N} is embedded in \mathbb{R}^2 such that ρ_N is a $\langle T, \varepsilon \rangle$ -sketch of G_{ρ_N} in \mathbb{R}^2 , then the span of G_{ρ_N} is less than $\frac{1}{2N} + \varepsilon$.*

Proof. In order to apply Lemma 2, we will produce an arc $W \subset [o, p_{|\alpha_N|-1}] \times [o, q_{|\beta_N|-1}]$. Intuitively, one may think of W as a pair of points travelling simultaneously, one on the leg $[o, p_{|\alpha_N|-1}]$ and the other on $[o, q_{|\beta_N|-1}]$, starting with both at o , ending with one at the end of its leg, and at every moment staying within distance $\frac{1}{2N} + \varepsilon$ from one another. With this in mind, and referring to Figure 1, one should be easily convinced that such a W may be defined which passes through the following pairs, in order: (o, o) , $(p_{n(0)}, q_{m(0)})$, $(p_{n(0)-\phi(0)}, q_{m(0)+\phi(0)})$, $(p_{n(0)}, q_{m(1)})$, $(p_{n(0)+\theta(0)}, q_{m(1)-\theta(0)})$, $(p_{n(1)}, q_{m(1)})$, \dots , $(p_{n(2N)}, q_{m(2N)})$, $(p_{|\alpha_N|-1}, q_{m(2N)+6N+5})$. The precise definition of this arc W follows.

Suppose that n, n' and m, m' are two pairs of adjacent integers. Let $S_{m,m'}^{n,n'}$ denote the square $[p_n, p_{n'}] \times [q_m, q_{m'}]$. Suppose one of the following occurs:

- (1) $w(p_n) = w(q_m)$, $w(p_{n'}) = w(q_{m'})$;
- (2) $w(p_n) = w(q_m)$, $w(p_{n'}) = b_{i/2N}$, $w(q_{m'}) = b_{(i+1)/2N}$ for some i ; or
- (3) $w(p_{n'}) = w(q_{m'})$, $w(p_n) = b_{i/2N}$, $w(q_m) = b_{(i+1)/2N}$ for some i .

Then let $W_{m,m'}^{n,n'} \subset S_{m,m'}^{n,n'}$ be an arc such that $d(x_1, x_2) < \frac{1}{2N} + \varepsilon$ for each $(x_1, x_2) \in W_{m,m'}^{n,n'}$, and $W_{m,m'}^{n,n'} \cap \partial S_{m,m'}^{n,n'} = \{(p_n, q_m), (p_{n'}, q_{m'})\}$.

Define the arc $W \subset [o, p_{|\alpha_N|-1}] \times [o, q_{|\beta_N|-1}]$ as follows. It will be helpful to refer to Figure 1 when reading this formula.

$$\begin{aligned}
W := & \bigcup_{j=0}^{n(0)-1} W_{j,j+1}^{j,j+1} \cup \bigcup_{i=0}^{2N-1} \left(\bigcup_{j=0}^{\phi(i)-1} W_{m(i)+j, m(i)+j+1}^{n(i)-j, n(i)-j-1} \cup \right. \\
& \bigcup_{j=0}^{\phi(i)-1} W_{m(i)+\phi(i)+j, m(i)+\phi(i)+j+1}^{n(i)-\phi(i)+j, n(i)-\phi(i)+j+1} \cup \\
& \bigcup_{j=0}^{\theta(i)-1} W_{m(i+1)-j, m(i+1)-j-1}^{n(i)+j, n(i)+j+1} \cup \\
& \left. \bigcup_{j=0}^{\theta(i)-1} W_{m(i+1)-\theta(i)+j, m(i+1)-\theta(i)+j+1}^{n(i)+\theta(i)+j, n(i)+\theta(i)+j+1} \right) \cup \\
& \bigcup_{j=0}^{6N+4} W_{m(2N)+j, m(2N)+j+1}^{n(2N)+j, n(2N)+j+1}.
\end{aligned}$$

Then W contains (o, o) and meets $\{p_{|\alpha_N|-1}\} \times [o, q_{|\beta_N|-1}]$, and $d(x_1, x_2) < \frac{1}{2N} + \varepsilon$ for each $(x_1, x_2) \in W$, hence the claim follows by Lemma 2. \square

3. COMBINATORICS FROM CHAIN COVERS

3.1. Chain quasi-orders. The following definition is closely related to the notion of a chain word reduction from [14]. It should be thought of as follows: if $\langle G, w \rangle$ is a $\langle T, \varepsilon \rangle$ -sketch of G and we have a chain cover of G of small mesh, then $v_1 \leq v_2$ means roughly that the chain ‘‘covers v_1 before, or at around the same time as, v_2 ’’ (see Proposition 5).

Definition. Suppose $\langle G, w \rangle$ is a compliant graph-word. A *chain quasi-order* of $\langle G, w \rangle$ is a total quasi-order \leq on $V(G)$ satisfying:

- (C1) if $v_1 \simeq v_2$, then $w(v_1) \approx_\Gamma w(v_2)$;
- (C2) if $v_1, v_2 \in V(G)$ are adjacent in G , then v_1 and v_2 are \leq -adjacent; and
- (C3) if $v_1, v_2, v_3 \in V(G)$ are consecutive in G , $v \in V(G)$, and if $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0, 1]$ are such that $t' \geq t$, $v_1 v_2 v_3 \xrightarrow{w} \sigma b_t \tau$, $w(v) = b_{t'}$, and $v_1 < v_2 \simeq v < v_3$, then $t' - t < \frac{1}{2}$.

Notice that if \leq is a chain quasi-order, then the reverse order of \leq (defined by $v_1 \leq^* v_2$ if and only if $v_2 \leq v_1$) is also a chain quasi-order.

The following simple lemma will be useful later on.

Lemma 4. *Let \leq be a chain quasi-order of $\langle G, w \rangle$. Suppose $v_1, s_1, \dots, s_\kappa, v_2$ are consecutive in G and $v \in V(G)$ is such that $v_1 < v < v_2$. Then there is some $i \in \{1, \dots, \kappa\}$ such that $v \simeq s_i$.*

Proof. Put $s_0 := v_1$, $s_{\kappa+1} := v_2$, and let i be the largest integer in $\{0, \dots, \kappa\}$ such that $s_i \leq v$. Then $s_{i+1} > v$, so since s_i and s_{i+1} are \leq -adjacent by property (C2), we must have $s_i \simeq v$. \square

3.2. Chain covers and the triod T_0 .

Proposition 5. *Suppose $\langle G, w \rangle$ is a compliant graph-word which is a $\langle T_0, \varepsilon \rangle$ -sketch of a graph G in \mathbb{R}^2 , where $0 < \varepsilon < \frac{1}{2}$. If there is a chain cover for G of mesh $< \frac{1}{2} - \varepsilon$, then there is a chain quasi-order of $\langle G, w \rangle$.*

Proof. Let $\mathcal{U} = \langle U_\ell : 0 \leq \ell < L \rangle$ be a chain cover for G of mesh $< \frac{1}{2} - \varepsilon$, ordered so that $U_\ell \cap U_{\ell'} \neq \emptyset$ if and only if $|\ell - \ell'| \leq 1$. For each $v \in V(G)$, let $\ell(v)$ be such that $v \in U_{\ell(v)}$ (for each v there are either one or two choices for $\ell(v)$).

Observe that if $v_1, v_2 \in V(G)$ and $\ell(v_1) = \ell(v_2)$, then $w(v_1) \approx_\Gamma w(v_2)$, since otherwise $d(\iota(w(v_1)), \iota(w(v_2))) \geq \sqrt{2} > \frac{1}{2}$, hence $d(v_1, v_2) > \frac{1}{2} - \varepsilon$, contradicting the condition that the diameter of $U_{\ell(v_1)} = U_{\ell(v_2)}$ is $< \frac{1}{2} - \varepsilon$.

Define the relation \leq on $V(G)$ by setting $v_1 \leq v_2$ if and only if for every $v \in V(G)$ satisfying $\ell(v_2) \leq \ell(v) \leq \ell(v_1)$ we have $w(v) \approx_\Gamma w(v_1)$.

The following properties follow directly from the definition of \leq :

- Properties.**
- (1) If $\ell(v_1) \leq \ell(v_2)$, then $v_1 \leq v_2$.
 - (2) If $v_1 \leq v_2$ and $w(v_1) \not\approx_\Gamma w(v_2)$, then $\ell(v_1) < \ell(v_2)$.
 - (3) If $v_1, v_2 \in V(G)$ are \leq -adjacent, then $w(v_1) \approx_\Gamma w(v_2)$.

It is straightforward to check using the definition and these properties that \leq is a total quasi-order.

We now check that \leq satisfies properties (C1), (C2), and (C3) of the definition of a chain quasi-order.

(C1): Suppose $v_1, v_2 \in V(G)$ with $v_1 \simeq v_2$. Assume without loss of generality that $\ell(v_2) \leq \ell(v_1)$. It then follows immediately from the definition of \leq and the assumption $v_1 \leq v_2$ that $w(v_1) \approx_\Gamma w(v_2)$ (take $v = v_2$).

(C2): Suppose $v_1, v_2 \in V(G)$ are adjacent in G . Since $\langle G, w \rangle$ is compliant, we know that $w(v_1) \not\approx_\Gamma w(v_2)$. Let $\sigma := w(v_1)$ and $\tau := w(v_2)$. Assume without loss of generality that $\ell(v_1) < \ell(v_2)$, which implies that $v_1 < v_2$.

If $v \in V(G)$ were such that $w(v) \not\approx_{\Gamma} \sigma, \tau$ and $v_1 < v < v_2$, then $\ell(v_1) < \ell(v) < \ell(v_2)$. This would imply that the link $U_{\iota(v)}$ contains the point v and meets the arc $[v_1, v_2]$. Since $\langle G, w \rangle$ is compliant, the only possible cases are:

$$\begin{aligned} \{\sigma, \tau\} &= \{a, b\} \text{ and } w(v) = c \\ \{\sigma, \tau\} &= \{a, c\} \text{ and } w(v) \in \{b\} \cup \{b_t : t \in [0, 1]\} \\ \{\sigma, \tau\} &= \{b, c\} \text{ and } w(v) = a \\ \{\sigma, \tau\} &= \{a, b_t\} \text{ and } w(v) = c && \text{(for some } t \in [0, 1]) \\ \{\sigma, \tau\} &= \{c, b_t\} \text{ and } w(v) = a && \text{(for some } t \in [0, 1]) \end{aligned}$$

In each case, we have $d(\iota(w(v)), [\iota(\sigma), \iota(\tau)]) \geq 1 > \frac{1}{2}$. But this yields a contradiction, since \mathcal{U} has mesh $< \frac{1}{2} - \varepsilon$.

Suppose for a contradiction that v_1, v_2 are not adjacent in the \leq order. Let v, v' be such that $v_1 < v < v'$, and v_1, v are \leq -adjacent and v, v' are \leq -adjacent. By the above, we have that $w(v), w(v')$ are each \approx_{Γ} to either σ or τ , hence by Property (3) the only possibility is $w(v) \approx_{\Gamma} \tau, w(v') \approx_{\Gamma} \sigma$. Property (2) then implies that $\ell(v_1) < \ell(v) < \ell(v')$.

Define the arc $A \subset T_0$ according to the value of σ as follows:

$$A := \begin{cases} [\iota(a), o] & \text{if } \sigma = a \\ [\iota(c), o] & \text{if } \sigma = c \\ [\iota(b), o] & \text{if } \sigma \in \{b\} \cup \{b_t : t \in [0, 1]\} \end{cases}.$$

In each case, observe that $d(\iota(w(v)), A) \geq 1 > \frac{1}{2}$, and also $B_{\frac{1}{2}}(\iota(\sigma)) \subset A$ and $B_{\frac{1}{2}}(\iota(w(v'))) \subset A$.

Applying the homeomorphism $\widehat{w}|_{[v_1, v_2]}$ yields the chain cover $\langle \widehat{w}(U_{\ell} \cap [v_1, v_2]) : \ell' \leq \ell \leq \ell'' \rangle$ of the arc $[\iota(\sigma), \iota(\tau)]_{T_0}$, where $\ell' := \min\{\ell : U_{\ell} \cap [v_1, v_2] \neq \emptyset\}$ and $\ell'' := \max\{\ell : U_{\ell} \cap [v_1, v_2] \neq \emptyset\}$.

Notice that $\widehat{w}(U_{\ell(v_1)})$ and $\widehat{w}(U_{\ell(v')})$ are sets of diameter $< \frac{1}{2}$ containing $\iota(\sigma)$ and $\iota(w(v'))$, respectively, hence are subsets of A . It follows in particular that the links $\widehat{w}(U_{\ell(v_1)} \cap [v_1, v_2])$ and $\widehat{w}(U_{\ell(v')} \cap [v_1, v_2])$ both meet the arc $A \cap [\iota(\sigma), \iota(\tau)]_{T_0}$, which implies each link $\widehat{w}(U_{\ell} \cap [v_1, v_2])$, $\ell(v_1) < \ell < \ell(v')$, must meet A as well. But $\widehat{w}(U_{\ell(v)})$ has diameter $< \frac{1}{2}$ and contains $\iota(w(v))$, hence misses A by the above. This is a contradiction, therefore we must have that v_1 and v_2 are \leq -adjacent.

(C3): Suppose $v \in V(G)$, v_1, v_2, v_3 are consecutive in G , and that $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0, 1]$ are such that $t' \geq t$, $w(v) = b_{t'}$, $v_1 v_2 v_3 \xrightarrow{w} \sigma b_t \tau$, and $v_1 < v_2 \simeq v < v_3$.

From Property (2) we know that $\ell(v)$ is between $\ell(v_1)$ and $\ell(v_3)$, hence the link $U_{\ell(v)}$ contains v and meets the arc $[v_1, v_2] \cup [v_2, v_3]$. Since $d(\iota(b_{t'}), [\iota(\sigma), \iota(b_t)]_{T_0} \cup [\iota(b_t), \iota(\tau)]_{T_0}) = d(\iota(b_{t'}), \iota(b_t)) = t' - t$ and \mathcal{U} has mesh $< \frac{1}{2} - \varepsilon$, it follows that $t' - t < \frac{1}{2}$. \square

4. COMBINATORICS OF THE GRAPH-WORD ρ_N

4.1. Chain quasi-orders and ρ_N . Throughout this subsection assume that $\langle G, w \rangle$ is a compliant graph-word, and that \leq is a chain quasi-order of $\langle G, w \rangle$.

Let $f : V(G) \rightarrow \mathbb{Z}$ be an order preserving function whose range is a contiguous block of integers.

Lemma 6. *Suppose v_1, \dots, v_n are consecutive in G , and that for each $1 < j < n$ we have $w(v_{j-1}) \not\approx_{\Gamma} w(v_{j+1})$. Then $f(v_1), \dots, f(v_n)$ are consecutive integers, i.e. either $f(v_{j+1}) = f(v_j) + 1$ for each $1 \leq j < n$, or $f(v_{j+1}) = f(v_j) - 1$ for each $1 \leq j < n$.*

Proof. This follows immediately from properties (C1) and (C2) of the chain quasi-order \leq . \square

As an application of Lemma 6, we make the following observation.

Lemma 7. *Suppose for some $i < 2N$ that $v_0, v_1, \dots, v_{2\theta(i)} \in V(G)$ are consecutive in G with $v_0 \cdots v_{2\theta(i)} \xrightarrow{w} \alpha_N(n(i)) \cdots \alpha_N(n(i+1))$. Let $k := f(v_0)$. Then we have one of the following four cases:*

- (1) $v_0 \cdots v_{2\theta(i)} \xrightarrow{f} k \cdots (k + 2\theta(i))$;
- (2) $v_0 \cdots v_{\theta(i)} \xrightarrow{f} k \cdots (k + \theta(i))$, $v_{\theta(i)} \cdots v_{2\theta(i)} \xrightarrow{f} (k + \theta(i)) \cdots k$;
- (3) $v_0 \cdots v_{\theta(i)} \xrightarrow{f} k \cdots (k - \theta(i))$, $v_{\theta(i)} \cdots v_{2\theta(i)} \xrightarrow{f} (k - \theta(i)) \cdots k$; or
- (4) $v_0 \cdots v_{2\theta(i)} \xrightarrow{f} k \cdots (k - 2\theta(i))$.

Moreover, the analogous statement holds for the word β_N^- (where we replace n with m and θ with ϕ).

Proof. This is a simple consequence of Lemma 6. \square

Lemma 8. *Suppose that for each $i \in \{0, N, 2N\}$, there are $v_1^{(i)}, v_2^{(i)}, v_3^{(i)} \in V(G)$ which are consecutive in G with $v_1^{(i)} v_2^{(i)} v_3^{(i)} \xrightarrow{w} ab_{i/2N}c$. Then it cannot be the case that $v_3^{(0)} \simeq v_3^{(N)} \simeq v_3^{(2N)}$.*

Proof. Suppose for a contradiction that $f(v_3^{(0)}) = f(v_3^{(N)}) = f(v_3^{(2N)}) = k$. By Lemma 6, for each $i \in \{0, N, 2N\}$ we have either

$$v_1^{(i)} v_2^{(i)} v_3^{(i)} \xrightarrow{f} (k-2)(k-1)k$$

or

$$v_1^{(i)} v_2^{(i)} v_3^{(i)} \xrightarrow{f} (k+2)(k+1)k.$$

It then follows from the pidgeonhole principle that $f(v_2^{(i)}) = f(v_2^{(j)})$ for distinct $i, j \in \{0, N, 2N\}$. But this contradicts property (C3) of the chain quasi-order \leq . \square

Lemma 9. *Suppose $v_0, \dots, v_{|\alpha_N|-1} \in V(G)$ are consecutive in G and $v'_0, \dots, v'_{|\beta_N|-2} \in V(G)$ are consecutive in G with $v_0 \cdots v_{|\alpha_N|-1} \xrightarrow{w} \alpha_N$ and $v'_0 \cdots v'_{|\beta_N|-2} \xrightarrow{w} \beta_N^-$. Suppose further that $v_0 \simeq v'_0$. Then $v_1 \not\approx v'_1$.*

Proof. Assume without loss of generality that $v_0 \leq v_1$. Suppose for a contradiction that $v_1 \simeq v'_1$.

We know that $f(v_0) \leq f(v_1)$ and that $f(v_0) = f(v'_0)$, $f(v_1) = f(v'_1)$. Put $k := f(v_{n(0)})$, and recall that $n(0) = 6N + 5 = m(0)$. It follows from Lemma 6 that

$$\begin{aligned} v_0 \cdots v_{n(0)} &\xrightarrow{f} (k - 6N - 5) \cdots k, \text{ and} \\ v'_0 \cdots v'_{m(0)} &\xrightarrow{f} (k - 6N - 5) \cdots k. \end{aligned}$$

Claim 9.1. Let $i < 2N$. If $f(v_{n(i)}) = k$ and $f(v_{n(i)+\theta(i)}) < k$, then $f(v_{n(i+1)}) = k$. Similarly, if $f(v'_{m(i)}) = k$ and $f(v'_{m(i)+\phi(i)}) < k$, then $f(v'_{m(i+1)}) = k$.

Proof of Claim 9.1. Suppose $f(v_{n(i)}) = k > f(v_{n(i)+\theta(i)})$. If

$$v_{n(i)} \cdots v_{n(i+1)} \xrightarrow{f} k \cdots (k - 2\theta(i)),$$

then in particular $f(v_{n(i)+\theta(i)+1}) = k - \theta(i) - 1$. Also, $f(v_{n(0)-\theta(i)-1}) = k - \theta(i) - 1$. But $w(v_{n(i)+\theta(i)+1}) = c \not\approx_{\Gamma} a = w(v_{n(0)-\theta(i)-1})$, so this contradicts property (C1) of the chain quasi-order \leq . Therefore by Lemma 7 we must have $f(v_{n(i+1)}) = k$.

Similarly, suppose $f(v'_{m(i)}) = k > f(v'_{m(i)+\phi(i)})$. If

$$v'_{m(i)} \cdots v'_{m(i+1)} \xrightarrow{f} k \cdots (k - 2\phi(i)),$$

then in particular $f(v'_{m(i)+\phi(i)+1}) = k - \phi(i) - 1$. Also, $f(v'_{m(0)-\phi(i)-1}) = k - \phi(i) - 1$. But $w(v_{m(i)+\phi(i)+1}) = b \not\approx_{\Gamma} c = w(v'_{m(0)-\phi(i)-1})$, so this contradicts property (C1) of the chain quasi-order \leq . Therefore by Lemma 7 we must have $f(v'_{m(i+1)}) = k$. \square (Claim 9.1)

Claim 9.2. Either $f(v_{n(i)}) = k$ for each $i \leq 2N$ or $f(v'_{m(i)}) = k$ for each $i \leq 2N$.

Proof of Claim 9.2. If $f(v_{n(i)+\theta(i)}) < k$ and $f(v'_{m(i)+\phi(i)}) < k$ for each $i < 2N$, then this follows immediately from Claim 9.1 and induction. Hence assume this is not the case, and let i^* be the smallest i for which $f(v_{n(i)+\theta(i)}) > k$ or $f(v'_{m(i)+\phi(i)}) > k$.

Observe that by Claim 9.1 and induction, we have $f(v_{n(i)}) = f(v'_{m(i)}) = k$ for each $i \leq i^*$.

Case 1. $f(v'_{m(i^*)+\phi(i^*)}) > k$.

It follows from Lemma 6 that

$$v'_{m(i^*)} \cdots v'_{m(i^*)+\phi(i^*)} \xrightarrow{f} k \cdots (k + \phi(i^*)).$$

Suppose $i^* \leq i < 2N$, and that $f(v_{n(i)}) = k$. If $f(v_{n(i)+\theta(i)}) < k$, then we have by Claim 9.1 that $f(v_{n(i+1)}) = k$.

If $f(v_{n(i)+\theta(i)}) > k$, suppose for a contradiction that

$$v_{n(i)} \cdots v_{n(i+1)} \xrightarrow{f} k \cdots (k + 2\theta(i)).$$

In particular, this means $f(v_{n(i)+\theta(i)+1}) = k + \theta(i) + 1$. Also, since $\phi(i^*) > \theta(i)$, we have $f(v'_{m(i^*)+\theta(i)+1}) = k + \theta(i) + 1$. But $w(v_{n(i)+\theta(i)+1}) = c \not\approx_{\Gamma} a = w(v'_{m(i^*)+\theta(i)+1})$, so this contradicts property (C1) of the chain quasi-order \leq . Therefore by Lemma 7 we must have $f(v_{n(i+1)}) = k$.

Thus by induction, we have $f(v_{n(i)}) = k$ for each $i \leq 2N$.

Case 2. $f(v'_{m(i^*)+\phi(i^*)}) < k$ and $f(v_{n(i^*)+\theta(i^*)}) > k$.

Here we have by Claim 9.1 that $f(v'_{m(i^*+1)}) = k$.

It follows from Lemma 6 that

$$v_{n(i^*)} \cdots v_{n(i^*)+\theta(i^*)} \xrightarrow{f} k \cdots (k + \theta(i^*)).$$

Suppose $i^* + 1 \leq i < 2N$, and that $f(v'_{m(i)}) = k$. If $f(v'_{m(i)+\phi(i)}) < k$, then we have by Claim 9.1 that $f(v'_{m(i+1)}) = k$.

If $f(v'_{m(i)+\phi(i)}) > k$, suppose for a contradiction that

$$v'_{m(i)} \cdots v'_{m(i+1)} \xrightarrow{f} k \cdots (k + 2\phi(i)).$$

In particular, this means $f(v'_{m(i)+\phi(i)+1}) = k + \phi(i) + 1$. Also, since $\theta(i^*) > \phi(i)$, we have $f(v_{n(i^*)+\phi(i)+1}) = k + \phi(i) + 1$. But $w(v'_{m(i)+\phi(i)+1}) = b \not\approx_{\Gamma} c = w(v_{n(i^*)+\phi(i)+1})$, so this contradicts property (C1) of the chain quasi-order \leq . Therefore by Lemma 7 we must have $f(v'_{m(i+1)}) = k$.

Thus by induction, we have $f(v'_{m(i)}) = k$ for each $i \leq 2N$. \square (Claim 9.2)

It remains only to notice that Claim 9.2 contradicts Lemma 8. So we must have $v_1 \not\approx v'_1$. \square

For convenience in later statements and arguments, we will use the following notation:

Definition. Given $\sigma \in \Gamma$, define the word $\zeta_N(\sigma)$ by

$$\zeta_N(\sigma) := \begin{cases} \alpha_N & \text{if } \sigma = a \\ \beta_N & \text{if } \sigma = b \\ \gamma_N & \text{if } \sigma = c \\ \beta_N^- b_t & \text{if } \sigma = b_t \text{ (for some } t \in [0, 1]). \end{cases}$$

Lemma 10. Suppose $\sigma, \tau \in \Gamma$, $v_0, \dots, v_{\kappa} \in V(G)$ are consecutive in G , and $v'_0, \dots, v'_{\lambda} \in V(G)$ are consecutive in G , with

$$v_0 \cdots v_{\kappa} \xrightarrow{w} \zeta_N(\sigma) \quad \text{and} \quad v'_0 \cdots v'_{\lambda} \xrightarrow{w} \zeta_N(\tau).$$

Suppose further that $v_0 \simeq v'_0$ and $v_1 \simeq v'_1$. Then $\sigma \approx_{\Gamma} \tau$.

Proof. Suppose for a contradiction that $\sigma \not\approx_{\Gamma} \tau$. If $\sigma = a$ and $\tau \in \{b, b_t : t \in [0, 1]\}$, or vice versa, then this contradicts Lemma 9. If one of them is c , say σ , then $w(v_1) = c$ while $w(v'_1) = b \not\approx_{\Gamma} c$, so this contradicts property (C1) of the chain quasi-order \leq . \square

Proposition 11. There is no chain quasi-order for ρ_N , for any N .

Proof. Suppose for a contradiction that \leq is a chain quasi-order for ρ_N . Observe that since $r, p_1, q_1 \in V(G_{\rho_N})$ are all adjacent to o in G_{ρ_N} , we have that these three vertices are also adjacent to o in the \leq order. Hence by the pidgeonhole principle, some pair of them are in the relation \simeq . But this is a contradiction by Lemma 10. \square

Oversteegen and Tymchatyn exhibit in [17] for each $\delta > 0$ a 2-dimensional plane strip with span $< \delta$ which has no chain cover of mesh < 1 . Repovš et al. modify this example in [21] to construct for each $\delta > 0$ a tree in the plane with span $< \delta$ which has no chain cover of mesh < 1 . In both examples, the diameters of the continua converge to ∞ as $\delta \rightarrow 0$. We pause to point out that we have now obtained a bounded family of such examples.

Corollary 12. There is a uniformly bounded sequence $\langle T_N \rangle_{N=1}^{\infty}$ of simple triods in \mathbb{R}^2 such that for each N , $\text{span}(T_N) < \frac{1}{N}$ and T_N has no chain cover of mesh $< \frac{1}{4}$.

Proof. This is simply a combination of Propositions 1 (using T_0 and taking $\varepsilon \leq \frac{1}{2N}$), 3, 5, and 11. \square

We are working to prove a stronger result: that there is a continuum in \mathbb{R}^2 which has span zero and cannot be covered by a chain of mesh less than some positive constant. To this end we will need some further technical combinatorial lemmas.

Lemma 13. *Suppose $\sigma, \tau \in \Gamma$ with $\sigma \approx_\Gamma \tau$, and that $v_0, \dots, v_\kappa \in V(G)$ are consecutive in G and $v'_0, \dots, v'_\kappa \in V(G)$ are consecutive in G with*

$$v_0 \cdots v_\kappa \xrightarrow{w} \zeta_N(\sigma) \quad \text{and} \quad v'_0 \cdots v'_\kappa \xrightarrow{w} \zeta_N(\tau).$$

Then:

- (i) if $v_0 < v_1$, then $v_0 < v_j < v_\kappa$ for each $0 < j < \kappa$;
- (ii) if $v_{\kappa-1} < v_\kappa$, then $v_0 < v_j < v_\kappa$ for each $0 < j < \kappa$;
- (iii) if $v_0 \simeq v'_0$ and $v_1 \simeq v'_1$, then $v_\kappa \simeq v'_\kappa$; and
- (iv) if $v_\kappa \simeq v'_\kappa$ and $v_{\kappa-1} \simeq v'_{\kappa-1}$, then $v_0 \simeq v'_0$.

Proof. Each of these statements is trivial if $\sigma = \tau = c$. We will prove the Lemma for $\sigma = \tau = a$; the case $\sigma \approx_\Gamma \tau \approx_\Gamma b$ proceeds analogously.

- (i) Suppose $v_0 < v_1$.

Claim 13.1. $v_0 \cdots v_{n(0)} \xrightarrow{f} (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)})$.

Proof of Claim 13.1. This is immediate from Lemma 6. \square (Claim 13.1)

Claim 13.2. For each $i < 2N$, $v_{n(i)} \leq v_{n(i+1)}$.

Proof of Claim 13.2. We proceed by induction on $i < 2N$. Suppose the claim is true for each i' with $i' < i$. Put $k := f(v_{n(i)})$. Suppose for a contradiction that $f(v_{n(i)}) > f(v_{n(i+1)})$. By Lemma 7, this means

$$v_{n(i)} \cdots v_{n(i+1)} \xrightarrow{f} k \cdots (k - 2\theta(i)).$$

In particular, we have $f(v_{n(i)+\theta(i)+1}) = k - \theta(i) - 1$.

Let j^* be the smallest $j \leq i$ such that $f(v_{n(j)}) = k$.

If $j^* = 0$, then since $n(0) > \theta(i)$, we have $f(v_{n(0)-\theta(i)-1}) = k - \theta(i) - 1$. But also $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v_{n(0)-\theta(i)-1})$, so this contradicts property (C1) of the chain quasi-order \leq .

If $j^* > 0$, then we know by Lemma 7 that

$$v_{n(j^*-1)} \cdots v_{n(j^*)} \xrightarrow{f} (k - 2\theta(j^* - 1)) \cdots k.$$

Then similarly observe that since $\theta(j^* - 1) > \theta(i)$, we have $f(v_{n(j^*)-\theta(i)-1}) = k - \theta(i) - 1$. But also $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v_{n(j^*)-\theta(i)-1})$, so this contradicts property (C1) of the chain quasi-order \leq . \square (Claim 13.2)

Claim 13.3. $v_{n(2N)} \cdots v_\kappa \xrightarrow{f} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5)$.

Proof of Claim 13.3. By Lemma 8 and Claim 13.2, we must have $v_{n(i-1)} < v_{n(i)}$ for some $0 < i \leq 2N$; let i^* be the largest such i , so that $f(v_{n(2N)}) = f(v_{n(i^*)})$.

Suppose for a contradiction that

$$v_{n(2N)} \cdots v_\kappa \xrightarrow{f} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) - 6N - 5).$$

Then in particular, since $6N + 5 > \theta(i^* - 1)$, we have $f(v_{n(2N)+\theta(i^*-1)+1}) = f(v_{n(2N)} - \theta(i^* - 1) - 1)$. But also $f(v_{n(i^*)-\theta(i^*-1)-1}) = f(v_{n(2N)} - \theta(i^* - 1) - 1)$ and $w(v_{n(i^*)-\theta(i^*-1)-1}) = c \not\approx_\Gamma a = w(v_{n(2N)+\theta(i^*-1)+1})$, so this

contradicts property (C1) of the chain quasi-order \leq . Therefore by Lemma 6, we must have

$$v_{n(2N)} \cdots v_\kappa \xrightarrow{f} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5).$$

□(Claim 13.3)

It is now easy to check that $f(v_0) = f(v_{n(0)}) - 6N - 5 < f(v_j) < f(v_{n(2N)}) + 6N + 5 = f(v_\kappa)$ for any $0 < j < \kappa$.

- (ii) Observe that if we consider the reverse order of \leq , part (i) gives that if $v_0 > v_1$, then $v_0 > v_j > v_\kappa$ for each $0 < j < \kappa$. In particular, this would mean $v_{\kappa-1} > v_\kappa$. Therefore if $v_{\kappa-1} < v_\kappa$ then $v_0 < v_1$, hence the conclusion follows from part (i).
- (iii) Suppose $v_0 \simeq v'_0$, $v_1 \simeq v'_1$, and assume without loss of generality that $v_0 < v_1$. This means Claims 13.1, 13.2, and 13.3 hold for the v_j 's and the v'_j 's. By Claim 13.1, we have

$$v_0 \cdots v_{n(0)} \xrightarrow{f} (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)})$$

and

$$v'_0 \cdots v'_{n(0)} \xrightarrow{f} (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)}).$$

Claim 13.4. For each $i \leq 2N$, $v_{n(i)} \simeq v'_{n(i)}$.

Proof of Claim 13.4. Suppose not, and let i^* be the smallest $i < 2N$ such that $v_{n(i+1)} \not\simeq v'_{n(i+1)}$. Put $k := f(v_{n(i^*)}) = f(v'_{n(i^*)})$. It follows from Lemma 7 and Claim 13.2 that either $f(v_{n(i^*+1)}) = k$ and $f(v'_{n(i^*+1)}) > k$, or $f(v_{n(i^*+1)}) > k$ and $f(v'_{n(i^*+1)}) = k$; assume the former. This implies by Lemma 7 that

$$v'_{n(i^*)} \cdots v'_{n(i^*+1)} \xrightarrow{f} k \cdots (k + 2\theta(i^*)).$$

We claim that $f(v_{n(i)}) = k$ for each $i \geq i^*$. Indeed, given $i > i^*$, suppose for a contradiction that

$$v_{n(i)} \cdots v_{n(i+1)} \xrightarrow{f} k \cdots (k + 2\theta(i)).$$

This means in particular that $f(v_{n(i)+\theta(i)+1}) = k + \theta(i) + 1$. Since $\theta(i) < \theta(i^*)$, we have $f(v'_{n(i^*)+\theta(i)+1}) = k + \theta(i) + 1$. But $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v'_{n(i^*)+\theta(i)+1})$, so this contradicts property (C1) of the chain quasi-order \leq . Therefore by Lemma 7 and Claim 13.2, we must have $f(v_{n(i+1)}) = k$. Hence, by induction, $f(v_{n(i)}) = k$ for each $i \geq i^*$.

In particular, $f(v_{n(2N)}) = k$. By Claim 13.3, we have

$$v_{n(2N)} \cdots v_\kappa \xrightarrow{f} k \cdots (k + 6N + 5).$$

Since $6N + 5 > \theta(i^*)$, this means that $f(v_{n(2N)+\theta(i^*)+1}) = k + \theta(i^*) + 1$. Note $f(v'_{n(i^*)+\theta(i^*)+1}) = k + \theta(i^*) + 1$ as well. But $w(v'_{n(i^*)+\theta(i^*)+1}) = c \not\approx_\Gamma a = w(v_{n(2N)+\theta(i^*)+1})$, so this contradicts property (C1) of the chain quasi-order \leq . □(Claim 13.4)

Claim 13.4 implies in particular that $f(v_{n(2N)}) = f(v'_{n(2N)})$. Then by Claim 13.3, we have

$$v_{n(2N)} \cdots v_\kappa \xrightarrow{f} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5),$$

and

$$v'_{n(2N)} \cdots v'_\kappa \xrightarrow{f} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5).$$

This establishes part (iii).

- (iv) Suppose $v_\kappa \simeq v'_\kappa$, $v_{\kappa-1} \simeq v'_{\kappa-1}$, and assume without loss of generality that $v_{\kappa-1} < v_\kappa$. By part (ii) this implies $v_0 < v_1$ and $v'_0 < v'_1$, so again Claims 13.1, 13.2, and 13.3 hold for the v_j 's and the v'_j 's. By Claim 13.3, we have

$$v_\kappa \cdots v_{n(2N)} \xrightarrow{f} (f(v_{n(2N)}) + 6N + 5) \cdots f(v_{n(2N)})$$

and

$$v'_\kappa \cdots v'_{n(2N)} \xrightarrow{f} (f(v_{n(2N)}) + 6N + 5) \cdots f(v_{n(2N)}).$$

Claim 13.5. For each $i \leq 2N$, $v_{n(i)} \simeq v'_{n(i)}$.

Proof of Claim 13.5. Suppose not, and let i^* be the largest $i < 2N$ such that $v_{n(i)} \not\simeq v'_{n(i)}$. Put $k := f(v_{n(i^*+1)}) = f(v'_{n(i^*+1)})$. It follows from Lemma 7 and Claim 13.2 that either $f(v_{n(i^*)}) = k$ and $f(v'_{n(i^*)}) < k$, or $f(v_{n(i^*)}) < k$ and $f(v'_{n(i^*)}) = k$; assume the former. This implies by Lemma 7 that

$$v'_{n(i^*+1)} \cdots v'_{n(i^*)} \xrightarrow{f} k \cdots (k - 2\theta(i^*)).$$

We claim that $f(v_{n(i)}) = k$ for each $i \leq i^*$. Indeed, given $i < i^*$, suppose for a contradiction that

$$v_{n(i+1)} \cdots v_{n(i)} \xrightarrow{f} k \cdots (k - 2\theta(i)).$$

Since $\theta(i^*) < \theta(i)$, this means in particular that $f(v_{n(i+1)-\theta(i^*)-1}) = k - \theta(i^*) - 1$. Note $f(v'_{n(i^*+1)-\theta(i^*)-1}) = k - \theta(i^*) - 1$ as well. But $w(v_{n(i^*+1)-\theta(i^*)-1}) = c \not\approx_\Gamma a = w(v'_{n(i^*+1)-\theta(i^*)-1})$, so this contradicts property (C1) of the chain quasi-order \leq . Therefore by Lemma 7 and Claim 13.2 we must have $f(v_{n(i)}) = k$. Hence, by induction, $f(v_{n(i)}) = k$ for each $i \leq i^*$.

In particular, $f(v_{n(0)}) = k$. By Claim 13.1, we have

$$v_{n(0)} \cdots v_0 \xrightarrow{f} k \cdots (k - 6N - 5).$$

Since $6N + 5 > \theta(i^*)$, this means that $f(v_{n(0)-\theta(i^*)-1}) = k - \theta(i^*) - 1$. Note $f(v'_{n(i^*+1)-\theta(i^*)-1}) = k - \theta(i^*) - 1$ as well. But $w(v'_{n(i^*+1)-\theta(i^*)-1}) = c \not\approx_\Gamma a = w(v_{n(0)-\theta(i^*)-1})$, so this contradicts property (C1) of the chain quasi-order \leq . \square (Claim 13.5)

Claim 13.5 implies in particular that $f(v_{n(0)}) = f(v'_{n(0)})$. Then by Claim 13.1, we have

$$v_{n(0)} \cdots v_0 \xrightarrow{f} f(v_{n(0)}) \cdots (f(v_{n(0)}) - 6N - 5)$$

and

$$v'_{n(0)} \cdots v'_0 \xrightarrow{f} f(v_{n(0)}) \cdots (f(v_{n(0)}) - 6N - 5).$$

This establishes part (iv). \square

4.2. Iterated sketches. If $\iota_T : \Gamma \rightarrow T$ is a Γ -marking of the simple triod T and ρ_N is a $\langle T, \varepsilon \rangle$ -sketch of the simple triod graph $T' := G_{\rho_N}$ such that $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{T'} = [\iota_T(c), \iota_T(b)]_T$ (as in Proposition 1), then one can define an induced Γ -marking $\iota_{T'} : \Gamma \rightarrow T'$ on T' as follows: define $\iota_{T'}(a) := p_{|\alpha_N|-1}$, $\iota_{T'}(b) := q_{|\beta_N|-1} = \iota_T(b)$, $\iota_{T'}(c) := r$, and for each $t \in [0, 1]$ put $\iota_{T'}(b_t) := \iota_T(b_t) \in [q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{T'} = [\iota_T(c), \iota_T(b)]_T$.

Now let T_0 be as before, and suppose T_1 and T_2 are simple triods such that ρ_1 is a $\langle T_0, \varepsilon_0 \rangle$ -sketch of T_1 , and ρ_2 is a $\langle T_1, \varepsilon_1 \rangle$ -sketch of T_2 (using the induced Γ -marking on T_1). Evidently we should be able to find a $\langle T_0, \varepsilon_0 + \varepsilon_1 \rangle$ -sketch of T_2 , and indeed this is necessary if we want to apply Proposition 5 to argue that T_2 has no chain cover of small mesh. This is the motivation for the next definition (see Proposition 14).

Definition. Suppose $\langle G, w \rangle$ is a compliant graph-word, and $N > 0$. A graph-word $\langle G^+, w^+ \rangle$ is a ρ_N -expansion of $\langle G, w \rangle$ if:

- G^+ is identical to G as a topological space, but the vertex set of G^+ is finer: for any adjacent pair of vertices $v_1, v_2 \in V(G)$, there are distinct degree 2 vertices $s_j^{v_1 v_2}$, $j = 1, \dots, \kappa_{v_1 v_2}$ where $\kappa_{v_1 v_2} = |\zeta_N(w(v_1))| + |\zeta_N(w(v_2))| - 3$, inserted into the edge joining v_1, v_2 so that $v_1, s_1^{v_1 v_2}, \dots, s_{\kappa_{v_1 v_2}}^{v_1 v_2}, v_2$ are consecutive in G^+ ; and
- w^+ is defined by

$$v_1 s_1^{v_1 v_2} \cdots s_{\kappa_{v_1 v_2}}^{v_1 v_2} v_2 \xrightarrow{w^+} \zeta_N(w(v_1))^\leftarrow \cap \zeta_N(w(v_2))$$

when $v_1, v_2 \in V(G)$ are adjacent in G .

Remarks. (1) Notice that $w^+|_{V(G)} = w$, and that $\langle G^+, w^+ \rangle$ is also a compliant graph-word.

- (2) Combinatorially, there is only one ρ_N expansion of a given graph-word $\langle G, w \rangle$; however, geometrically they may differ according to where along the edges of G the extra vertices are inserted (though their order on the edge is determined uniquely by the definition).

Proposition 14. *Let T be a Γ -marked simple triod. Suppose ρ_N is a $\langle T, \varepsilon_1 \rangle$ -sketch of $T' := G_{\rho_N}$, and that $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{T'} = [\iota_T(c), \iota_T(b)]_T$ (as in Proposition 1). Endow T' with a Γ -marking as in the beginning of subsection 4.2. If $\rho = \langle G, w \rangle$ is a compliant graph-word which is a $\langle T', \varepsilon_2 \rangle$ -sketch of G , then there is a ρ_N -expansion of $\langle G, w \rangle$ which is a $\langle T, \varepsilon_1 + \varepsilon_2 \rangle$ -sketch of G .*

Proof. Let $\widehat{w}_{\rho_N} : T' \rightarrow T$ be a ρ_N -suggested bonding map such that $d(x, \widehat{w}_{\rho_N}(x)) < \frac{\varepsilon_1}{2}$ for each $x \in T'$, and let $\widehat{w} : G \rightarrow T'$ be a ρ -suggested bonding map such that $d(x, \widehat{w}(x)) < \frac{\varepsilon_2}{2}$ for each $x \in G$.

Consider any adjacent $v_1, v_2 \in V(G)$. Define

$$s_i^{v_1 v_2} := \begin{cases} (\widehat{w}|_{[v_1, v_2]})^{-1}(p_{|\alpha_N|-1-i}) & \text{if } w(v_1) = a \\ (\widehat{w}|_{[v_1, v_2]})^{-1}(q_{|\beta_N|-1-i}) & \text{if } w(v_1) \approx_{\Gamma} b \\ (\widehat{w}|_{[v_1, v_2]})^{-1}(o) & \text{if } w(v_1) = c \end{cases}$$

for $1 \leq i \leq |\zeta_N(w(v_1))| - 1$, and

$$s_{\kappa_{v_1 v_2} + 1 - i}^{v_1 v_2} := \begin{cases} (\widehat{w}|_{[v_1, v_2]})^{-1}(p_{|\alpha_N|-1-i}) & \text{if } w(v_2) = a \\ (\widehat{w}|_{[v_1, v_2]})^{-1}(q_{|\beta_N|-1-i}) & \text{if } w(v_2) \approx_{\Gamma} b \end{cases}$$

for $1 \leq i \leq |\zeta_N(w(v_2))| - 2$.

Let $V(G^+)$ be equal to $V(G)$ together with all these new vertices, and let w^+ be defined as in the definition of a ρ_N -expansion. Observe that $w^+ = w_{\rho_N} \circ (\widehat{w}|_{V(G^+)})$. Put $\rho^+ := \langle G^+, w^+ \rangle$, where G^+ is equal to G as a topological space, with vertex set $V(G^+)$.

It is now straightforward to see that $\widehat{w_{\rho_N}} \circ \widehat{w}$ is a ρ^+ -suggested bonding map, and clearly $d(x, (\widehat{w_{\rho_N}} \circ \widehat{w})(x)) < \frac{\varepsilon_1 + \varepsilon_2}{2}$ for each $x \in G$. \square

Lemma 15. *Suppose $\langle G, w \rangle$ is a compliant graph-word, let $\langle G^+, w^+ \rangle$ be a ρ_N -expansion of $\langle G, w \rangle$, and suppose \leq^+ is a chain quasi-order of $\langle G^+, w^+ \rangle$.*

- (i) *Let $v_1, v_2 \in V(G)$ be adjacent in G , and let $s_1, \dots, s_\kappa \in V(G^+) \setminus V(G)$ be such that $v_1, s_1, \dots, s_\kappa, v_2$ are consecutive in G^+ . Then the following are equivalent:*
- (1) $v_1 <^+ v_2$;
 - (2) $v_1 <^+ s_j <^+ v_2$ for each $j \in \{1, \dots, \kappa\}$;
 - (3) $v_1 <^+ s_j <^+ v_2$ for some $j \in \{1, \dots, \kappa\}$.
- (ii) *If $v_1, v_2 \in V(G)$ are adjacent in G and $v'_1, v'_2 \in V(G)$ are adjacent in G with $v_1 \simeq^+ v'_1$, $v_1 <^+ v_2$, and $v'_1 <^+ v'_2$, then $v_2 \simeq^+ v'_2$.*

Proof. (i) The implications (2) \Rightarrow (3) and (3) \Rightarrow (1) are trivial. For (1) \Rightarrow (2) we will prove that $v_1 <^+ s_1$ implies that $v_1 <^+ s_j <^+ v_2$ for each $j \in \{1, \dots, \kappa\}$. Then by considering the reverse order of \leq^+ , it follows that $v_1 <^+ v_2$ implies $v_1 <^+ s_1$, hence $v_1 <^+ s_j <^+ v_2$ for each $j \in \{1, \dots, \kappa\}$.

Suppose $v_1 <^+ s_1$. Let $i \in \{1, \dots, \kappa\}$ be such that

$$s_i \cdots s_1 v_1 \xrightarrow{w^+} \zeta_N(w(v_1)) \quad \text{and} \quad s_i \cdots s_\kappa v_2 \xrightarrow{w^+} \zeta_N(w(v_2)).$$

By Lemma 13 (ii), we have $v_1 <^+ s_j <^+ s_i$ for each $j \in \{1, \dots, i-1\}$. Because G is compliant, we can deduce using Lemma 10 that $s_i <^+ s_{i+1}$. Then by Lemma 13 (i) we have $s_i <^+ s_j <^+ v_2$ for each $j \in \{i+1, \dots, \kappa\}$.

- (ii) Suppose $v_1, v_2 \in V(G)$ are adjacent in G and $v'_1, v'_2 \in V(G)$ are adjacent in G with $v_1 \simeq^+ v'_1$, $v_1 <^+ v_2$, and $v'_1 <^+ v'_2$. Let s_1, \dots, s_κ and i be as in part (i), and let $s'_1, \dots, s'_\lambda \in V(G^+) \setminus V(G)$ be such that $v'_1 s'_1, \dots, s'_\lambda v'_2$ are consecutive in G^+ and

$$v'_1 s'_1 \cdots s'_\lambda v'_2 \xrightarrow{w^+} \zeta_N(w(v'_1)) \leftarrow \cap \zeta_N(w(v'_2)).$$

By property (C1) of the chain quasi-order \leq^+ , $w(v_1) \approx_\Gamma w(v'_1)$, hence $|\zeta_N(v_1)| = |\zeta_N(v'_1)|$, and so

$$s'_i \cdots s'_1 v'_1 \xrightarrow{w^+} \zeta_N(w(v'_1)) \quad \text{and} \quad s'_i \cdots s'_\lambda v'_2 \xrightarrow{w^+} \zeta_N(w(v'_2)).$$

By Lemma 13 (iv), we have $s_i \simeq^+ s'_i$, and as in part (i) we know that $s'_{i+1} >^+ s'_i$. By Lemma 10, this implies $w(v_2) \approx_\Gamma w(v'_2)$, hence $\kappa = \lambda$. Then by Lemma 13 (iii), we conclude that $v_2 \simeq^+ v'_2$. \square

Proposition 16. *Suppose $\langle G, w \rangle$ is a compliant graph-word. If a (any) ρ_N -expansion of $\langle G, w \rangle$ has a chain quasi-order, then $\langle G, w \rangle$ also has a chain quasi-order.*

Proof. Let $\langle G^+, w^+ \rangle$ be a ρ_N -expansion of $\langle G, w \rangle$, and let \leq^+ be a chain quasi-order of $\langle G^+, w^+ \rangle$.

Define \leq on $V(G)$ by $\leq := \leq^+|_{V(G)}$. Clearly \leq is a total quasi-order since \leq^+ is. We must check that \leq satisfies properties (C1), (C2), and (C3) of the definition of a chain quasi-order.

(C1): This is immediate since \leq^+ satisfies this property.

(C2): We will need the following claim:

Claim 16.1. In $\langle G^+, w^+ \rangle$, if $v \in V(G)$ and $v' \in V(G^+)$ are such that $v \simeq^+ v'$, then in fact $v' \in V(G)$.

Proof of Claim 16.1. We proceed by induction on the number of vertices in G .

If $|V(G)| = 1$, then there is nothing to prove.

Assume the claim holds for all such graph-words whose graph has n or fewer vertices, and assume $|V(G)| = n + 1$. Let $u \in V(G)$ be such that the subgraph G^- obtained by removing the vertex u (and all edges emanating from u) is connected. There is a ρ_N -expansion of $\langle G^-, w|_{V(G) \setminus \{u\}} \rangle$ which is a sub-graph-word of $\langle G^+, w^+ \rangle$ (it has vertex set $V(G^+) \cap G^-$), and the restriction of \leq^+ to this sub-graph-word is a chain quasi-order. By induction, the claim holds for G^- .

Let $u' \in V(G) \setminus \{u\}$ be adjacent to u in G . Let $s_1, \dots, s_\kappa \in V(G^+) \setminus V(G)$ be such that $u', s_1, \dots, s_\kappa, u$ are consecutive in G^+ and

$$u' s_1 \cdots s_\kappa u \xrightarrow{w^+} \zeta_N(w(u'))^{\leftarrow} \pitchfork \zeta_N(w(u)).$$

Assume $u' <^+ u$ (the other case proceeds similarly), which implies by Lemma 15 (i) that $u' <^+ s_j <^+ u$ for each $j \in \{1, \dots, \kappa\}$.

We have four things to check:

- (1) for each $y \in V(G) \setminus \{u\}$ and each $s \in V(G^+) \setminus V(G)$ in the ρ_N -expansion of G^- , $y \not\leq^+ s$;
- (2) for each $y \in V(G) \setminus \{u\}$ and each $j \in \{1, \dots, \kappa\}$, $y \not\leq^+ s_j$;
- (3) for each $s \in V(G^+) \setminus V(G)$ in the ρ_N -expansion of G^- , $u \not\leq^+ s$; and
- (4) for each $j \in \{1, \dots, \kappa\}$, $u \not\leq^+ s_j$.

Observe that (1) holds by induction, and (4) is immediate from the observation that $u' <^+ s_j <^+ u$ for each $j \in \{1, \dots, \kappa\}$. For (2) and (3), we consider two cases.

Case 1. For every $y \in V(G) \setminus \{u\}$, $y \leq^+ u'$.

Since $u' <^+ s_j <^+ u$ for each $j \in \{1, \dots, \kappa\}$, we have immediately that $y \not\leq^+ s_j$ for any $y \in V(G) \setminus \{u\}$.

Also, from Lemma 15 (i) it follows that for every $s \in V(G^+) \setminus V(G)$ in the ρ_N -expansion of G^- , $s <^+ u'$. Therefore $u \not\leq^+ s$ for any such s .

Case 2. There exists some $y \in V(G) \setminus \{u\}$ such that $u' <^+ y$.

Let \mathcal{P} be a path of vertices in G^- starting at u' and ending at y . Let y_1 be the latest vertex y' in \mathcal{P} with $y' \leq^+ u'$, and let y_2 be the next vertex in \mathcal{P} after y_1 , so that y_1 and y_2 are adjacent in G and $y_1 \leq^+ u' <^+ y_2$.

Suppose for a contradiction that $y_1 <^+ u'$. Let $z_1, \dots, z_\lambda \in V(G^+) \setminus V(G)$ be such that $y_1, z_1, \dots, z_\lambda, y_2$ are consecutive in G^+ . Then by Lemma 4 there is some $i \in \{1, \dots, \lambda\}$ such that $u' \simeq^+ z_i$. But the claim holds for G^- by induction, so this is a contradiction. Therefore we must have $u' \simeq^+ y_1$.

Then from Lemma 15 (ii) we know that $u \simeq^+ y_2$. It follows immediately that $u \not\leq^+ s$ for each $s \in V(G^+) \setminus V(G)$ in the ρ_N -expansion of G^- , because $y_2 \not\leq^+ s$ for every such s by induction.

Moreover, for each $j \in \{1, \dots, \kappa\}$, since $y_1 \simeq^+ u' <^+ s_j <^+ u \simeq^+ y_2$, we know from Lemma 4 that there is some $s \in V(G^+) \setminus V(G)$ inserted between y_1 and y_2 such that $s_j \simeq^+ s$. It follows that $y \not\simeq^+ s_j$ for any $y \in V(G) \setminus \{u\}$, because $y \not\simeq^+ s$ for every such y by induction. \square (Claim 16.1)

Now suppose $v_1, v_2 \in V(G)$ are adjacent in G , and assume $v_1 \leq v_2$. Let $s_1, \dots, s_\kappa \in V(G^+) \setminus V(G)$ be such that $v_1, s_1, \dots, s_\kappa, v_2$ are consecutive in $V(G^+)$. If $v \in V(G)$ were such that $v_1 < v < v_2$, then $v_1 <^+ v <^+ v_2$ as well, so by Lemma 4 there would be some $i \in \{1, \dots, \kappa\}$ such that $v \simeq^+ s_i$. But this contradicts Claim 16.1.

(C3): Suppose $v \in V(G)$, v_1, v_2, v_3 are consecutive in G , and that $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0, 1]$ are such that $t' \geq t$, $w(v) = b_{t'}$, $v_1 v_2 v_3 \xrightarrow{w} \sigma b_t \tau$, and $v_1 < v_2 \simeq v < v_3$.

Let $s_1, \dots, s_\kappa, s'_1, \dots, s'_\lambda \in V(G^+) \setminus V(G)$ be such that $v_1, s_1, \dots, s_\kappa, v_2, s'_\lambda, \dots, s'_1, v_3$ are consecutive in G^+ , and

$$v_1 s_1 \cdots s_\kappa v_2 s'_\lambda \cdots s'_1 v_3 \xrightarrow{w^+} \zeta_N(\sigma)^{\leftarrow} \pitchfork \beta_N^- b_t (\beta_N^-)^{\leftarrow} \pitchfork \zeta_N(\tau).$$

Observe that $w^+(s_\kappa) = w^+(s'_\lambda) = c$. Since $v_1 <^+ v_2$, by Lemma 15 (i) we must have $s_\kappa <^+ v_2$. Likewise, we have $v_2 <^+ s'_\lambda$. It now follows from property (C3) of the chain quasi-order \leq^+ that $t' - t < \frac{1}{2}$. \square

5. THE EXAMPLE

Example. There exists a continuum $X \subset \mathbb{R}^2$ which is non-chainable and has span zero.

Proof. First we define by recursion a sequence $\langle T_N \rangle_{N=0}^\infty$ of Γ -marked simple triods in \mathbb{R}^2 and a sequence $\langle \varepsilon_N \rangle_{N=0}^\infty$ of positive real numbers as follows.

Let $T_0 \subset \mathbb{R}^2$ be as defined in subsection 2.1, and put $\varepsilon_0 := \frac{1}{8}$.

Suppose T_N, ε_N have been defined. Apply Proposition 1 to obtain an embedding T_{N+1} of the simple triod graph $G_{\rho_{N+1}}$ in \mathbb{R}^2 such that ρ_{N+1} is a $\langle T_N, \varepsilon_N \rangle$ -sketch of T_{N+1} , and $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{T_{N+1}} = [\iota_{T_N}(c), \iota_{T_N}(b)]_{T_N}$. Endow T_{N+1} with a Γ -marking as in the beginning of subsection 4.2. Notice that $T_{N+1} \subset (T_N)_{\varepsilon_N}$, where Y_ε denotes the ε -neighborhood of the space Y in \mathbb{R}^2 . By Proposition 3, the span of T_{N+1} is $< \frac{1}{2(N+1)} + \varepsilon_N$. Let $0 < \varepsilon_{N+1} < 2^{-N-4}$ be small enough so that $\overline{(T_{N+1})_{\varepsilon_{N+1}}} \subseteq \overline{(T_N)_{\varepsilon_N}}$, and so that $\text{span}(\overline{(T_{N+1})_{\varepsilon_{N+1}}}) < \frac{1}{2(N+1)} + 2\varepsilon_N$.

Put $X := \bigcap_{N=0}^\infty \overline{(T_N)_{\varepsilon_N}}$.

Observe that for any N , we have $X \subseteq \overline{(T_{N+1})_{\varepsilon_{N+1}}}$, hence

$$\text{span}(X) \leq \text{span}(\overline{(T_{N+1})_{\varepsilon_{N+1}}}) < \frac{1}{2(N+1)} + 2\varepsilon_N.$$

Since ε_N converges to 0 as $N \rightarrow \infty$, it follows that X has span zero.

Suppose for a contradiction that X has a chain cover of mesh $< \frac{1}{4}$. Then there is some $N > 0$ for which T_N has a chain cover of mesh $< \frac{1}{4}$.

Define by recursion the graph-words $\langle G_i, w_i \rangle$, $0 \leq i \leq N-1$, as follows: $\langle G_{N-1}, w_{N-1} \rangle := \rho_N$, and for $i < N-1$, $\langle G_i, w_i \rangle$ is the ρ_{i+1} -expansion of $\langle G_{i+1}, w_{i+1} \rangle$ provided by Proposition 14 which is a $\langle T_i, \sum_{j=i}^{N-1} \varepsilon_j \rangle$ -sketch of T_N . In particular, $\langle G_0, w_0 \rangle$ is a $\langle T_0, \sum_{j=0}^{N-1} \varepsilon_j \rangle$ -sketch of T_N .

Since $\sum_{j=0}^{N-1} \varepsilon_j < \sum_{j=0}^{N-1} 2^{-j-3} < \frac{1}{4}$, by Proposition 5 we have that $\langle G_0, w_0 \rangle$ has a chain quasi-order. Then by Proposition 16 and induction, we obtain a chain

quasi-order for each graph-word $\langle G_i, w_i \rangle$. In particular, $\langle G_{N-1}, w_{N-1} \rangle$ has a chain quasi-order. But $\langle G_{N-1}, w_{N-1} \rangle$ is ρ_N , so this contradicts Proposition 11. \square

6. QUESTIONS

The construction presented in this paper can be carried out so that every proper subcontinuum of the resulting space is an arc; hence, in particular, it is far from being hereditarily indecomposable. On the other hand, it follows from results of [17] that if there exists a non-degenerate homogeneous continuum in the plane which is not homeomorphic to the circle, the pseudo-arc, or the circle of pseudo-arcs, then there would be one which is hereditarily indecomposable and with span zero. Given that the pseudo-arc is the only hereditarily indecomposable chainable continuum [2], this raises the following question:

Question 1 (See Problem 9 of [18]). *Is there a hereditarily indecomposable non-chainable continuum with span zero?*

If such an example exists, then by [19, Corollary 6] it would be a continuous image of the pseudo-arc. Since any map to a hereditarily indecomposable continuum is confluent [23, Lemma 15], it would also be a counterexample to Problem 84 of [5], which asks whether every confluent image of a chainable continuum is chainable.

Regarding the planarity of the example in this paper, while every chainable continuum can be embedded in the plane [2], the same is not known to be true of continua with span zero.

Question 2. *Can every continuum with span zero be embedded in \mathbb{R}^2 ?*

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