MATH 5067, Summer 2018

Singular homology course notes

Preface

These course notes are intended to be read and used as a reference by an instructor for a introductory course in singular homology. These notes aim to provide an introduction to the subject, guided by these goals:

(a) To provide a complete development of just enough singular homology theory to obtain some of the classical applications in topology, e.g. the Brouwer Fixed Point Theorem, the Jordan Curve Theorem, the Fundamental Theorem of Algebra, the Borsuk-Ulam Theorem, the covering dimension of $\mathbb{R}^m$, among others; and

(b) To rely on topological/visual ideas in favor of algebraic abstractions as much as possible.

I have attempted to avoid some common obstacles in learning homology, including:

- **Heavy use of abstract algebra.** For those who are enthusiastic about algebra, this may not be an obstacle. Indeed there is a beautiful and rich field of algebra which sprung from the original development of homology in topology, and now has a life of its own. Most standard modern texts on homology theory develop some of the abstract algebraic theory and apply it to deduce some results about singular homology. These notes exist to provide an alternative, where we stay grounded in singular homology throughout, only mentioning briefly the algeraic generalizations in an optional section.

  One advantage of this approach is that all the maps in the Mayer-Vietoris sequence (and the long exact sequence for the pair in the optional section) are concretely defined and can be visualized, and the proofs of various parts can be illustrated with informative pictures. In the algebraic approach, the definition of the "boundary" map is supressed, and usually one simply deduces the existence of such a map without explicitly describing it. In practice when applying the Mayer-Vietoris sequence, we seldom need the definition of that map to compute the homology groups up to isomorphism, but it can be used to find explicit generators.

- **Abundance of notation and terminology.** I have tried to economize as much as possible on notation and terminology in these notes. In particular, I avoided using (outside of optional sections) several common terms like n-boundary, homologous, chain complex, chain map, chain homotopy, CW-complex. I also present only one variant of homology, namely the reduced singular homology, supressing any mention of unreduced singular homology (except in optional sections), relative homology, homology subordinate to a cover, and simplicial homology.

  It seems to be a popular opinion that simplicial homology ought to be introduced to beginning students before singular homology, because those groups can be computed directly from their definitions. Moreover, in computational applications in combinatorial topology or data science, simplicial homology is essential. By skipping simplicial homology (as is done in these notes), an instructor asks the students to digest the definition of singular homology and prove properties of these groups (induced homomorphisms and the Mayer-Vietoris sequence) before seeing a single non-trivial example worked out. My experience is that students can accept this if warned well enough. Some advantages are that there is no need to discuss Δ-complexes/simplicial complexes, we don't make singular homology seem unduly intimidating by comparison, and we take a more direct route to the classical applications to topology.

  For an instructor who wishes to follow the route of starting with simplicial homology, this topic is very well-presented in most texts on homology.

- **Lack of pictures.** Most of the basic ideas of singular homology can be illustrated faithfully with informative pictures. I have included several in these notes.

My main reference in developing these notes was Allen Hatcher's book *Algebraic Topology*.

Prerequisites

Only a basic grasp of topology fundamentals (including continuous functions, compactness, and path connectedness) is required for this course, and a little bit of algebra with abelian groups (which is reviewed in the preliminaries section). The definitions of homotopy and homotopy equivalence are given in the preliminaries section. Students need not be familiar with the fundamental group of a space.

Notation

- In a generic topological space X, points are denoted by ordinary lower case letters: x
- Points in Euclidean space are decorated with an arrow: $\vec{x}$

  Exception: the vertices v_i of the standard simplex Δ^n.
- Singular n-simplices are denoted by lower-case Greek letters: σ, τ, ρ, etc.
- n-chains are denoted with bold letters: $\mathbf{x}$

Quotient groups

When working with a quotient group ${}^G/_H$ outside of a pure algebra course, I generally subscribe to notation that suggests that the elements of ${}^G/_H$ are really just the elements that we had in G, only in the quotient we consider some elements to be the same that weren't the same back in G. Specifically, given two elements g_1 and g_2 in G, we have that g_1 equals g_2 in the quotient ${}^G/_H$ if and only if $g_1 - g_2 \in H$. This is written "$g_1 \equiv g_2$ in ${}^G/_H$" in these notes. This is analagous to the notation often used for modular arithmetic on the integers.

A popular alternative is the more formally correct approach which emphasizes that elements of the quotient ${}^G/_H$ are equivalence classes of elements in G. Elements of ${}^G/_H$ then look like $[g]$ or $\overline{g}$ or $g+H$. An instructor who prefers such notation should have no difficulty translating to their preferred form.

Induced homomorphisms

A continuous function $f : X \to Y$ induces homomorphisms $f_\sharp : C_n(X) \to C_n(Y)$ and $f_* : \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$. According to the above comments on quotient groups, given $\mathbf{x} \in \widetilde{H}_n(X)$, I tend to think of $\mathbf{x}$ just as well as an element of $C_n(X)$, and to think of $f_\sharp(\mathbf{x})$ and $f_*(\mathbf{x})$ as being the same thing. To avoid awkwardness, I don't use the $f_\sharp$ notation at all once it is established that f_* is a well-defined function $\widetilde{H}_n(X) \to \widetilde{H}_n(Y)$. The only exception is in the proof of the Borsuk-Ulam theorem, where it becomes useful to keep these functions apart.

Pictures

I draw several pictures involving singular n-simplices and n-chains. Usually the pictures for $n = 1$ are the most insightful.

- For $n = 0$, the pictures are completely faithful. Pictures involving 0-cycles are usually more meaningful when the space X has more than one path component.
- For $n = 1$, a single static picture of the range of a singular 1-simplex cannot convey all the information about its parameterization. An arrow is drawn to indicate the orientation, which is enough to ascertain the boundary.

  The singular 1-simplices in my pictures are mostly one-to-one functions, and generally look smooth. More "extreme" examples of singular 1-simplices, such as space filling curves or constant functions, cannot be conveyed well by these pictures.

  I seldom have two 1-simplices in the same picture intersect other than in their endpoints.
- For $n = 2$, again a single static picture of the range of a singular 2-simplex cannot convey all the information about its parameterization.

  The singular 2-simplices in my pictures are always one-to-one functions, and have sharp corners to help identify where the vertices v_0, v_1, v_2 of the standard 2-simplex are mapped. A curly arrow is drawn to indicate which vertex goes where: the corner closest to the tail of the arrow is the image of v_0; the next vertex in the direction indicated is the image of v_1, and the last vertex is the image of v_2. This is enough to ascertain the boundary.

Singular 2-simplices which are not one-to-one cannot be conveyed well by these pictures.

I seldom have two 2-simplices in the same picture intersect other than along an edge or in a single vertex.

- I do not attempt to draw pictures with singular n-simplices for $n \geq 3$.
- To represent the unique singular -1-simplex, I simply draw the symbol $\varnothing$ in the corner of the picture.

For a picture representing an n-chain $\mathbf{x} = \sum_j m_j \sigma_j$, each simplex σ_j is labelled with its integer coefficient m_j. These simplices are all drawn in the same color to indicate they are part of the same n-chain.

When I subdivide simplices in pictures, the subdivision I illustrate seldom resembles the one which would be obtained using iterated barycentric subdivision (in which some orientations/signs may be reversed, and more subdivision than "necessary" may occur).

0 Preliminaries

Given $m \geq 0$, $\mathbb{D}^m = \{\vec{x} \in \mathbb{R}^m : \|\vec{x}\| \leq 1\}$ and $\mathbb{S}^m = \{\vec{x} \in \mathbb{R}^{m+1} : \|\vec{x}\| = 1\}$. Note $\mathrm{Bd}(\mathbb{D}^m) = \mathbb{S}^{m-1}$.

$\mathbb{R}\mathrm{P}^m = {}^{\mathbb{S}^m}/_{\sim}$ where $\vec{x} \sim \vec{y}$ if and only if $\vec{x} = \pm\vec{y}$.

$\mathbb{S}^1 \vee \mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1$ (k times) is the bouquet of k circles, i.e. k circles all joined together at a single point, which are otherwise pairwise disjoint.

0.1 Algebra basics

All groups will be abelian with operation $+$ and identity 0. We write $G = 0$ to mean G is the trivial group whose only element is 0.

If S is a set, the **free abelian group** generated by S is the set $\langle S \rangle$ of all finite formal sums $\sum_j m_j s_j$, where $m_j \in \mathbb{Z}$ and $s_j \in S$. If $S = \{s_1, \ldots, s_k\}$ is finite, then $\langle S \rangle = \langle s_1, \ldots, s_k \rangle = \{\sum_{j=1}^k m_j s_j : m_j \in \mathbb{Z}\} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (k times).

Given a group H, any function $f : S \to H$ defines a homomorphism from $\langle S \rangle$ to H by linear extension $f\left(\sum_j m_j s_j\right) = \sum_j m_j f(s_j)$.

Given a homomorphism $f : G \to H$,

$$\mathrm{Ker}(f) = \{g \in G : f(g) = 0\}$$
$$\mathrm{Im}(f) = \{f(g) : g \in G\}$$

Recall that f is one-to-one if and only if $\mathrm{Ker}(f) = 0$.

If H is a subgroup of G (all subgroups are normal since these are abelian groups), the **quotient group** ${}^{G}/_{H}$ formally consists of equivalence classes $[g] = \{g + h : h \in H\}$ of elements in G with operation $[g_1] + [g_2] = [g_1 + g_2]$. By abuse of notation, instead of $[g]$ we write g, and instead of $[g_1] = [g_2]$ we write $g_1 \equiv g_2$ in ${}^{G}/_{H}$ (by analogy with modular arithmetic notation e.g. $11 \equiv 23 \bmod 12$).

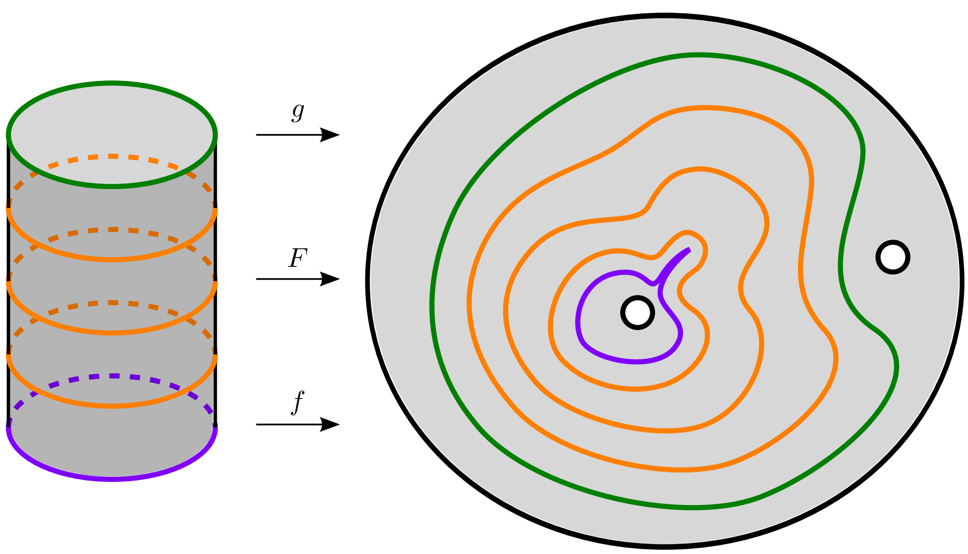
Given groups G_α, $\alpha \in I$, the **direct sum** is the group

$$\bigoplus_{\alpha \in I} G_\alpha = \{(g_\alpha)_{\alpha \in I} : g_\alpha \in G_\alpha \text{ for each } \alpha \in I \text{ and } g_\alpha = 0 \text{ for all but finitely many } \alpha\}$$

with coordinate-wise addition. If I is finite, then $\bigoplus_{\alpha \in I} G_\alpha = \prod_{\alpha \in I} G_\alpha$.

0.2 Homotopy

Definition. Functions $f, g : X \to Y$ are **homotopic** (in Y) if there exists a continuous function $F : X \times [0,1] \to Y$ (a **homotopy**) such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. We write $f \simeq g$.



Proposition 1. $\simeq$ is an equivalence relation (on functions).

Definition. X and Y are **homotopy equivalent** if there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We write $X \simeq Y$.

Proposition 2. $\simeq$ is an equivalence relation (on spaces).

X is **contractible** if it is homotopy equivalent to a point.

0.3 Exercises

Exercise 1. Observe that the circle $\mathbb{S}^1$, the annulus, the Möbius band, and the punctured plane $\mathbb{R}^2 \smallsetminus \{0\}$ are all homotopy equivalent.

Exercise 2. Show that $\mathbb{S}^1 \vee \mathbb{S}^1 \simeq \Theta$ (the Theta space).

Exercise 3. Observe that $\mathbb{D}^m$ is contractible.

Exercise 4. Let X and Y be topological spaces and assume Y is contractible. Prove that any two continuous functions from X to Y are homotopic.

Exercise 5. Let X be a topological space and let $f : \mathbb{S}^m \to X$ be continuous. Show that f is homotopic to a constant function $\mathbb{S}^m \to X$ if and only if f can be extended to a continuous function $\overline{f} : \mathbb{D}^{m+1} \to X$ (i.e. so that $\overline{f}|_{\mathrm{Bd}(\mathbb{D}^{m+1})} = f$).

Exercise 6. Let X be a topological space and let $m \geq 0$. Prove that if $f : X \to \mathbb{S}^m$ is not onto, then f is homotopic to a constant function.

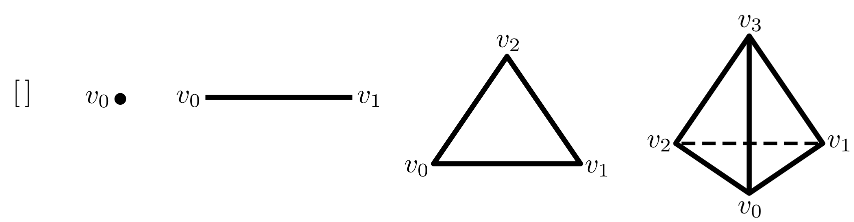
Exercise 7. Let X be a topological space and let $m \geq 0$. Let $f, g : X \to \mathbb{S}^m$ be continuous functions, and suppose $f(x) \neq -g(x)$ for all $x \in X$. Show that $f \simeq g$.

1 Singular homology

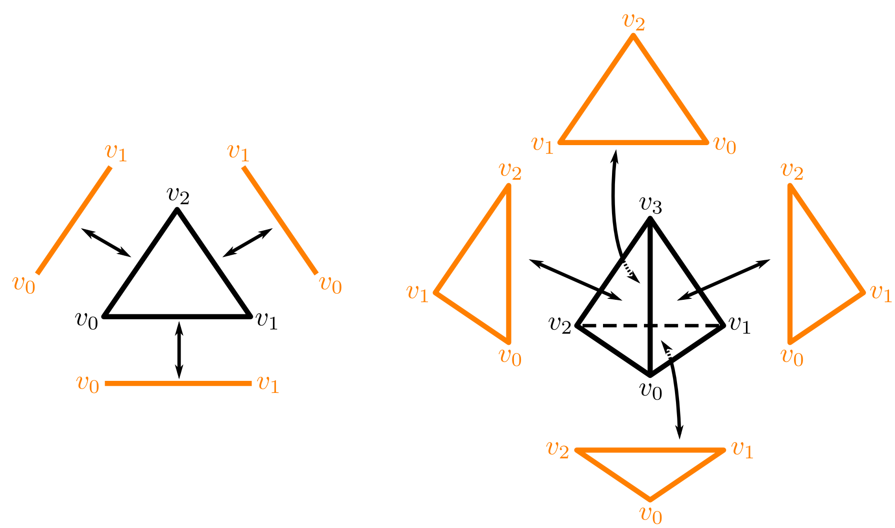
1.1 Definition of homology

Standard n-simplex: $\Delta^n = [v_0, \ldots, v_n]$ where $v_i = (0, \ldots, 1, \ldots, 0)$ (1 in $(i+1)$-st place). Formally

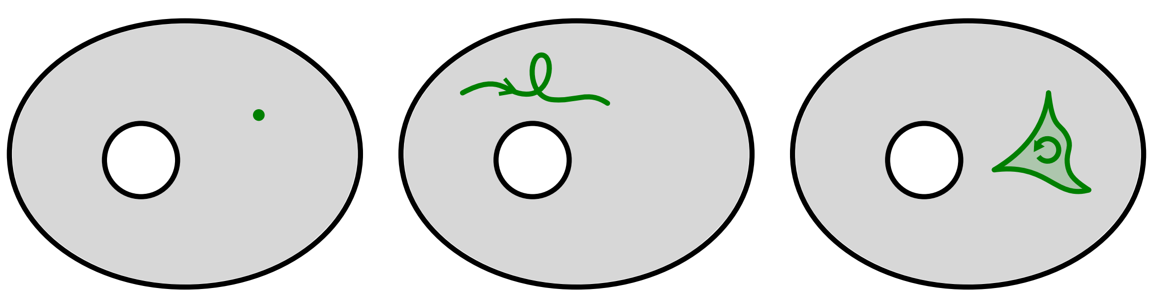
$$\Delta^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \right\}.$$



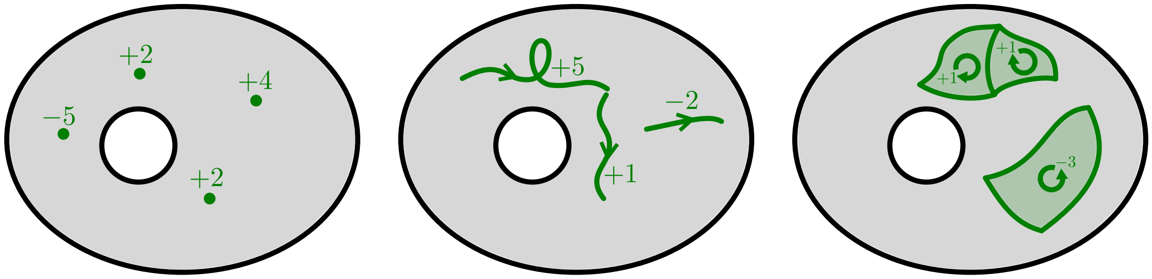
Face of Δ^n: $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$ (v_i removed). Identify with standard $(n-1)$-simplex Δ^{n-1} by canonical order-preserving linear isomorphism.



Fix a space X (non-empty).
A **singular n-simplex** in X is a continuous function $\sigma : \Delta^n \to X$.



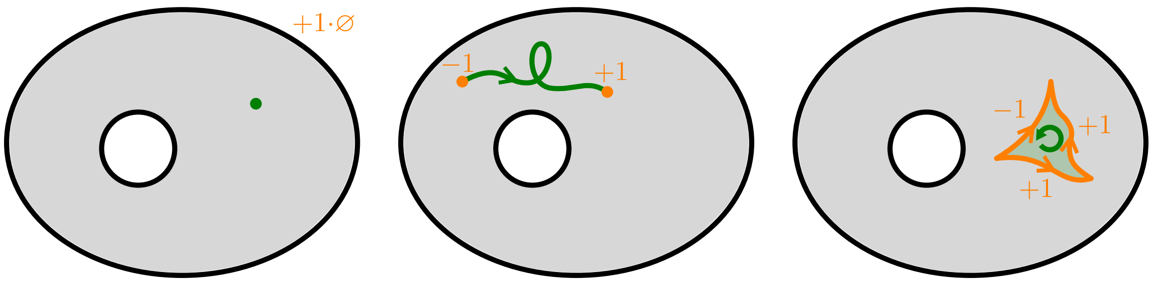
An ***n*-chain** in X is a (finite) (formal) sum of singular n-simplices $\mathbf{x} = \sum_j m_j \sigma_j$, where $m_j \in \mathbb{Z}$. $C_n(X) = \{n\text{-chains in } X\}$ (free abelian group generated by all singular n-simplices).



Technicality: We will say there is a standard -1-simplex, which is empty: $\Delta^{-1} = [\,]$. Correspondingly, there is a single singular -1-simplex in X, the empty function $\varnothing$. Thus $C_{-1}(X) = \langle \varnothing \rangle \cong \mathbb{Z}$.

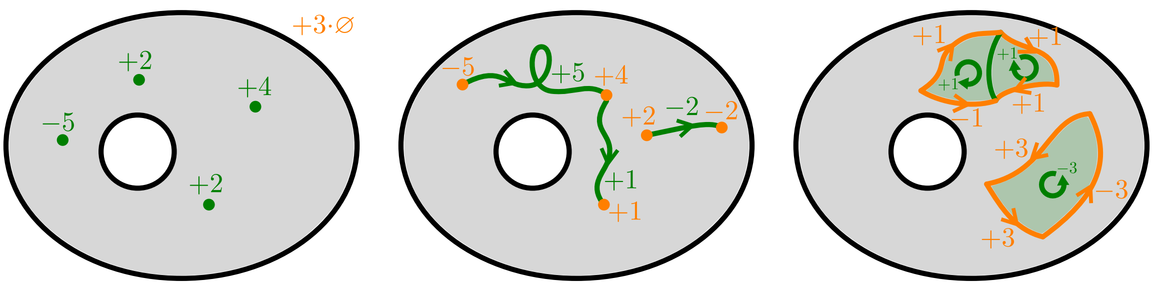
$C_n(X) = 0$ for all $n \leq -2$.

Boundary of an n-simplex σ: $\partial\sigma = \sum_{i=0}^{n}(-1)^i \sigma|[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$.

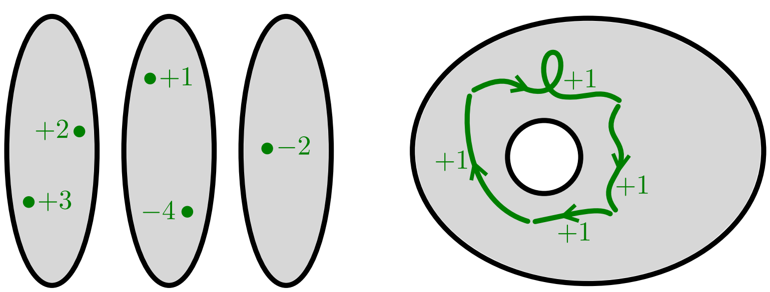


Technicality: For $n = -1$: $\partial\varnothing = 0$.

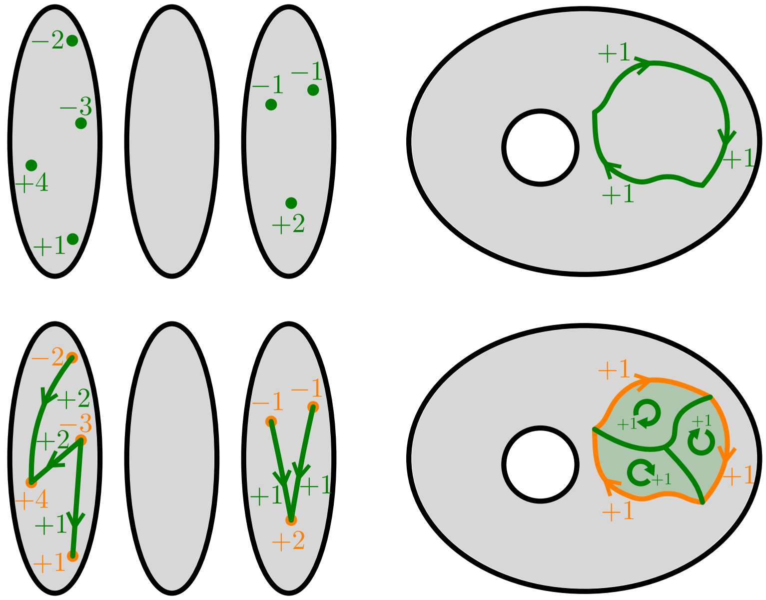
Boundary of an n-chain $\mathbf{x} = \sum_j m_j \sigma_j$: $\partial\mathbf{x} = \sum_j m_j \partial\sigma_j$. $\partial = \partial_n$ is a homomorphism $C_n(X) \to C_{n-1}(X)$.



An n-chain $\mathbf{x}$ is an ***n*-cycle** if $\partial\mathbf{x} = 0$. $\{n\text{-cycles}\} = \mathrm{Ker}(\partial_n)$.

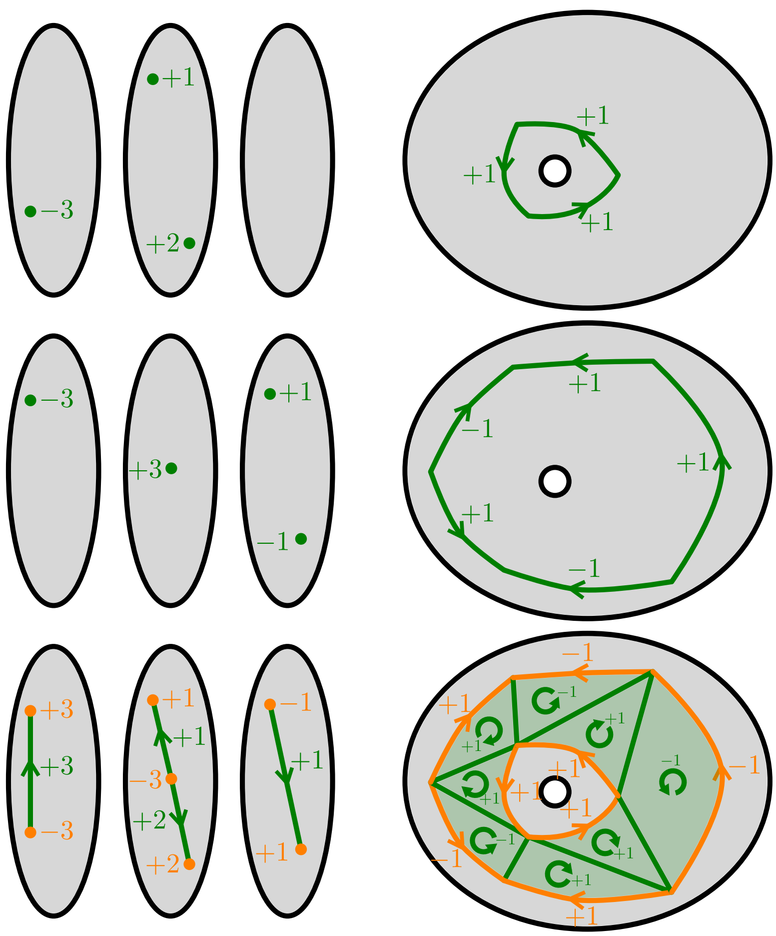


Idea: An n-cycle $\mathbf{x}$ may represent a (n-dimensional) hole in X, but not if $\mathbf{x}$ is the boundary of an $(n+1)$-chain. That is, if $\mathbf{x} \in \mathrm{Im}(\partial_{n+1})$, we set $\mathbf{x} \equiv 0$.



(Reduced) Homology group $\widetilde{H}_n(X) = {}^{\mathrm{Ker}(\partial_n)}/_{\mathrm{Im}(\partial_{n+1})}$.

Elements of $\widetilde{H}_n(X)$ are n-cycles, where two n-cycles $\mathbf{x}_1, \mathbf{x}_2$ are equal ($\equiv$) in $\widetilde{H}_n(X)$ if $\mathbf{x}_1 - \mathbf{x}_2$ is the boundary of some $(n+1)$-chain.



Lemma 1. (To confirm definition of $\widetilde{H}_n(X)$ makes sense) $\mathrm{Im}(\partial_{n+1}) \subseteq \mathrm{Ker}(\partial_n)$.

Proof. Let $\mathbf{x} = \sum_j m_j \sigma_j \in C_{n+1}(X)$.

$$\begin{aligned}
\partial_n \partial_{n+1} \mathbf{x} = \partial\partial\mathbf{x} = \partial \sum_j m_j \partial\sigma \\
&= \partial \sum_j m_j \sum_{i=0}^{n} (-1)^i \sigma|[v_0, \ldots, \widehat{v_i}, \ldots, v_n] \\
&= \sum_j m_j \sum_{i=0}^{n} (-1)^i \partial(\sigma|[v_0, \ldots, \widehat{v_i}, \ldots, v_n]) \\
&= \sum_j m_j \sum_{i=0}^{n} \left(\sum_{k=0}^{i-1} (-1)^{i+k} \sigma|[v_0, \ldots, \widehat{v_k}, \ldots, \widehat{v_i}, \ldots, v_n] + \sum_{k=i+1}^{n} (-1)^{i+k-1} \sigma|[v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_k}, \ldots, v_n] \right) \\
&= \sum_j m_j \cdot 0 \\
&= 0.
\end{aligned}$$

□

1.2 Basic observations

Observe $\widetilde{H}_n(X) = 0$ for all $n \leq -1$. For $n = -1$, one has $\{-1\text{-cycles}\} = C_{-1}(X) = \langle \varnothing \rangle$, and given any -1-cycle $m \cdot \varnothing$, we have $m \cdot \varnothing = \partial(m \cdot p)$, where p is any point of X (viewed as a 0-simplex), thus $\mathrm{Im}(\partial_0) = C_{-1}(X)$.

Theorem 2. If X has path components X_α, $\alpha \in I$, then $\widetilde{H}_n(X) \cong \bigoplus_{\alpha \in I} \widetilde{H}_n(X_\alpha)$ for all $n \geq 1$.

Theorem 3. If X has k path components, then $\widetilde{H}_0(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ ($k-1$ times). Specifically, if $p_1, \ldots, p_k \in X$ are any points from each path component, then $\widetilde{H}_0(X) = \langle p_2 - p_1, \ldots, p_k - p_1 \rangle$ (viewing p_i here as singular 0-simplices).

Example. Let $*$ be the space with just one point. Then $\widetilde{H}_n(*) = 0$ for all n.

Example. $\widetilde{H}_n(\mathbb{S}^0) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$

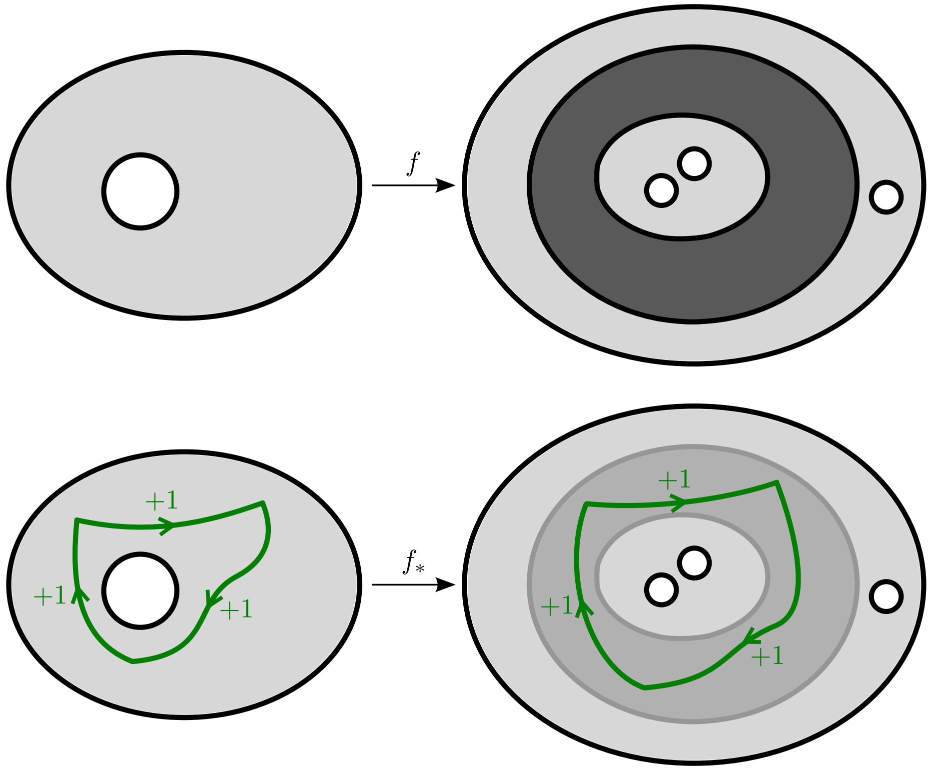
$\widetilde{H}_0(\mathbb{S}^0)$ is generated by $\mathbf{s}^0 = p - q$, where $p = -1 \in \mathbb{S}^0$ and $q = 1 \in \mathbb{S}^0$.

1.3 Induced homomorphisms

Suppose $f : X \to Y$ is a continuous function.

Given an n-simplex σ in X, $f \circ \sigma$ is an n-simplex in Y.

Define the **induced homomorphism** $f_* : \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ by $f_*(\sum_j m_j \sigma_j) = \sum_j m_j \cdot (f \circ \sigma_j)$.



Lemma 4. f_* is well defined.

Proof. The formula $\sum_j m_j \sigma_j \mapsto \sum_j m_j \cdot (f \circ \sigma_j)$ defines a function $f_\sharp : C_n(X) \to C_n(Y)$.

Claim 4.1. For any $\mathbf{x} \in C_n(X)$, $\quad f_\sharp(\partial \mathbf{x}) = \partial f_\sharp(\mathbf{x})$.

Proof of Claim 4.1. Given $\mathbf{x} = \sum_j m_j \sigma_j$,

$$\begin{aligned}
f_\sharp(\partial \mathbf{x}) &= f_\sharp\left(\partial \sum_j m_j \sigma_j\right) \\
&= f_\sharp\left(\sum_j m_j \sum_{i=0}^{n} (-1)^i \sigma_j|[v_0, \ldots, \widehat{v_i}, \ldots, v_n]\right) \\
&= \sum_j m_j \sum_{i=0}^{n} (-1)^i f \circ (\sigma_j|[v_0, \ldots, \widehat{v_i}, \ldots, v_n]) \\
&= \sum_j m_j \sum_{i=0}^{n} (-1)^i (f \circ \sigma_j)|[v_0, \ldots, \widehat{v_i}, \ldots, v_n] \\
&= \partial \sum_j m_j \cdot (f \circ \sigma_j) \\
&= \partial f_\sharp(\mathbf{x})
\end{aligned}$$

□(Claim 4.1)

To show f_* is well defined, there are two things to check:

- If $\mathbf{x} \in \widetilde{H}_n(X)$, this means $\partial \mathbf{x} = 0$, so $\partial f_\sharp(\mathbf{x}) = f_\sharp(\partial \mathbf{x}) = f_\sharp(0) = 0$. Thus $f_*(\mathbf{x})$ really is in $\widetilde{H}_n(Y)$.
- If $\mathbf{x}_1 \equiv \mathbf{x}_2$ in $\widetilde{H}_n(X)$, then $\mathbf{x}_1 - \mathbf{x}_2 = \partial \mathbf{x}_3$ for some $\mathbf{x}_3 \in C_{n+1}(X)$. Then $f_\sharp(\mathbf{x}_1) - f_\sharp(\mathbf{x}_2) = f_\sharp(\mathbf{x}_1 - \mathbf{x}_2) = f_\sharp(\partial \mathbf{x}_3) = \partial f_\sharp(\mathbf{x}_3)$. Thus $f_*(\mathbf{x}_1) \equiv f_*(\mathbf{x}_2)$ in $\widetilde{H}_n(Y)$.

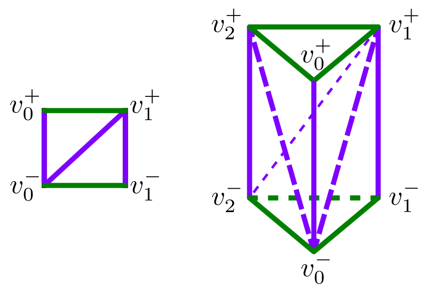
□

Theorem 5.
- $\mathrm{id}_* = \mathrm{id}$
- $(f \circ g)_* = f_* \circ g_*$ (whenever $g : W \to X$ and $f : X \to Y$).

Theorem 6. If $f \simeq g$, then $f_* = g_*$.

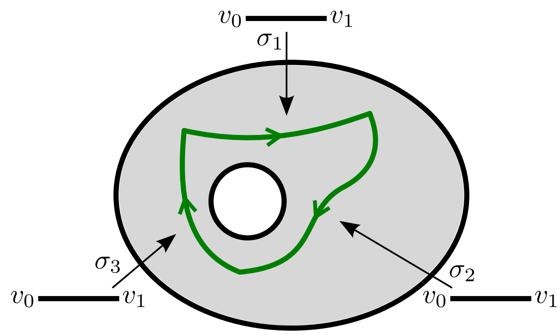
Proof. Let $F : X \times [0,1] \to Y$ be a homotopy between f and g, so $F|(X \times \{0\}) = f$ and $F|(X \times \{1\}) = g$.

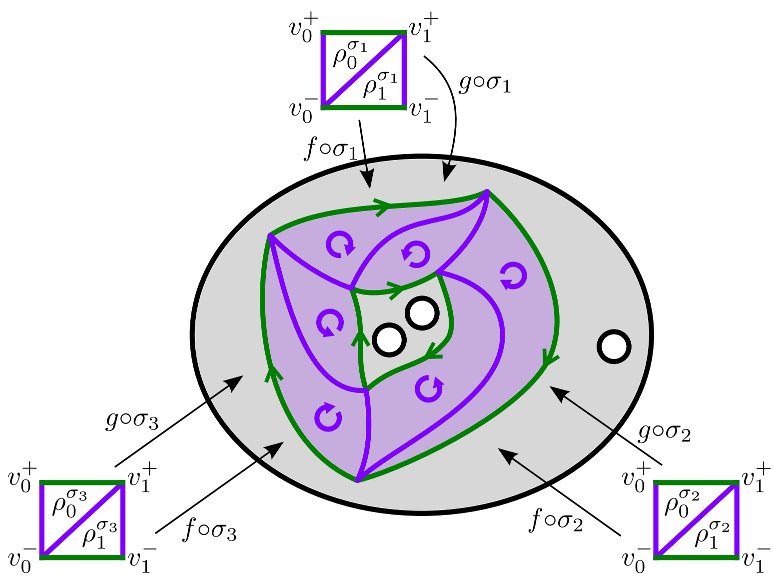
Given a singular n-simplex $\sigma : \Delta^n \to X$, we define an $(n+1)$-chain $\mathbf{P}(\sigma)$ in Y, to interpolate between $f \circ \sigma$ and $g \circ \sigma$, using the homotopy F as follows. In $\Delta^n \times I$, define $v_i^- = (v_i, 0)$ and $v_i^+ = (v_i, 1)$. For $k = 0, \ldots, n$, define $\Delta_k = [v_0^-, \ldots, v_k^-, v_k^+, \ldots, v_n^+]$, which we identify with Δ^{n+1} (via canonical order-preserving linear isomorphism). This partitions $\Delta^n \times [0,1]$ into $n+1$ $(n+1)$-simplices. Define $\rho_k^\sigma : \Delta_k \to Y$ by $\rho_k^\sigma(s,t) = F(\sigma(s), t)$. We treat ρ_k^σ as a singular $(n+1)$-simplex in Y. Define $\mathbf{P}(\sigma) = \sum_{k=0}^n (-1)^k \rho_k^\sigma \in C_{n+1}(Y)$.



Given $\mathbf{x} = \sum_j m_j \sigma_j \in C_n(X)$, define $\mathbf{P}(\mathbf{x}) = \sum_j m_j \mathbf{P}(\sigma_j)$. Now $\mathbf{P} = \mathbf{P}_n$ is a homomorphism $C_n(X) \to C_{n+1}(Y)$ for each n.

Let $\mathbf{x} = \sum_j m_j \sigma_j \in \widetilde{H}_n(X)$. To prove $f_*(\mathbf{x}) \equiv g_*(\mathbf{x})$ in $\widetilde{H}_n(Y)$, we will prove that $g_*(\mathbf{x}) - f_*(\mathbf{x}) = \partial \mathbf{P}(\mathbf{x})$.





Claim 6.1. Given a singular n-simplex $\sigma : \Delta^n \to X$, $\partial\mathbf{P}(\sigma) = g \circ \sigma - f \circ \sigma + \mathbf{P}(\partial\sigma)$.

Proof of Claim 6.1.

$$\begin{aligned}\partial\mathbf{P}(\sigma) &= \sum_{k=0}^{n}(-1)^k\partial\rho_k^\sigma \\ &= \sum_{k=0}^{n}\left(\sum_{i=0}^{k}(-1)^{i+k}\rho_k^\sigma|[v_0^-,\dots,\widehat{v_i^-},\dots,v_k^-,v_k^+,\dots,v_n^+] + \sum_{i=k}^{n}(-1)^{i+k+1}\rho_k^\sigma|[v_0^-,\dots,v_k^-,v_k^+,\dots,\widehat{v_i^+},\dots,v_n^+]\right)\end{aligned}$$

The term $k=0, i=0$ first sum is $g \circ \sigma$, and the term $k=n, i=n$ second sum is $-f \circ \sigma$.
For each $\ell = 0, \dots, n-1$, the terms $k=\ell, i=\ell$ second sum and $k=\ell+1, i=\ell+1$ first sum cancel.
The rest of the terms (ones where $k \neq i$) comprise $\mathbf{P}(\partial\sigma)$:

$$\begin{aligned}\mathbf{P}(\partial\sigma) &= \mathbf{P}\left(\sum_{i=0}^{n}(-1)^i\sigma|[v_0,\dots,\widehat{v_i},\dots,v_n]\right) \\ &= \sum_{i=0}^{n}\left(\sum_{k=0}^{i-1}(-1)^{i+k}\rho_k^\sigma|[v_0^-,\dots,v_k^-,v_k^+,\dots,\widehat{v_i^+},\dots,v_n^+] + \sum_{k=i+1}^{n}(-1)^{i+k+1}\rho_k^\sigma|[v_0^-,\dots,\widehat{v_i^-},\dots,v_k^-,v_k^+,\dots,v_n^+]\right)\end{aligned}$$

□(Claim 6.1)

Now $\partial\mathbf{P}(\mathbf{x}) = \sum_j m_j\partial\mathbf{P}(\sigma_j) = \sum_j m_j(g\circ\sigma_j - f\circ\sigma_j + \mathbf{P}(\partial\sigma_j)) = g_*(\mathbf{x}) - f_*(\mathbf{x}) + \mathbf{P}(\partial\mathbf{x}) = g_*(\mathbf{x}) - f_*(\mathbf{x}) + \mathbf{P}(0) = g_*(\mathbf{x}) - f_*(\mathbf{x})$. □

Corollary 7. If $X \simeq Y$, then $\widetilde{H}_n(X) \cong \widetilde{H}_n(Y)$ for all n.

Example. For $m \geq 0$, $\widetilde{H}_n(\mathbb{D}^m) = 0$ for all n.

1.4 Mayer-Vietoris sequence

Barycentric subdivision

All we need from this subsection going forward is the statement of the Subdivision Lemma below. The sketch of the proof of that lemma can be skipped if desired.

Let $\mathcal{U}$ be an open cover of X. An n-chain $\mathbf{x} = \sum_j m_j \sigma_j$ is **subordinate to $\mathcal{U}$** if for each j there exists $U_j \in \mathcal{U}$ such that $\sigma_j \subseteq U_j$.

Subdivision Lemma. Let $\mathcal{U}$ be an open cover of X, and let $\mathbf{x} \in C_n(X)$. If $\partial \mathbf{x}$ is subordinate to $\mathcal{U}$, then there exists $\mathbf{x}' \in C_n(X)$ such that $\mathbf{x}'$ is subordinate to $\mathcal{U}$ and $\mathbf{x} - \mathbf{x}' = \partial \mathbf{z}$ for some $\mathbf{z} \in C_{n+1}(X)$.

In particular, if $\mathbf{x} \in \widetilde{H}_n(X)$, then there exists $\mathbf{x}' \in \widetilde{H}_n(X)$ such that $\mathbf{x}'$ is subordinate to $\mathcal{U}$ and $\mathbf{x} \equiv \mathbf{x}'$ in $\widetilde{H}_n(X)$.

Sketch of proof. Given a (linear) n-simplex

$$\Delta = [w_0, \ldots, w_n] = \left\{ \sum_{i=0}^{n} t_i w_i \in \mathbb{R}^m : \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for each } i \right\}$$

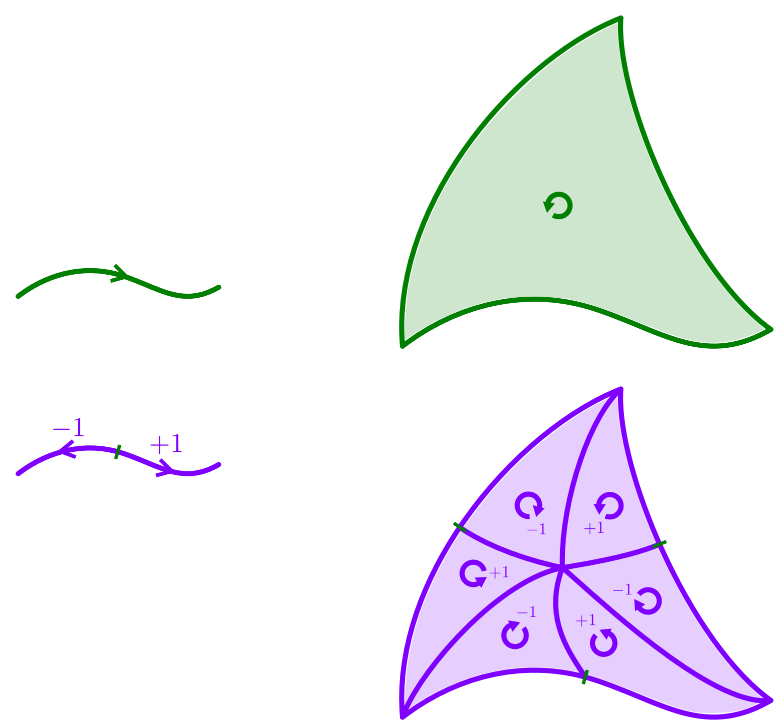
The **barycenter** is $b_\Delta = \sum_{i=0}^{n} \frac{1}{n+1} w_i$.

For $n \geq 1$, let $\sigma : \Delta \to X$ be a singular n-simplex, where $\Delta = [w_0, \ldots, w_n]$. Given a (linear) $(n-1)$-simplex $\Delta' = [w'_0, \ldots, w'_{n-1}]$ contained in a face of Δ, define $\mathbf{b}_\Delta(\sigma|\Delta') = \sigma|[b_\Delta, w'_0, \ldots, w'_{n-1}]$. Given finitely many such Δ'_j and integers m_j, define $\mathbf{b}_\Delta(\sum_j m_j(\sigma|\Delta'_j)) = \sum_j m_j \mathbf{b}_\Delta(\sigma|\Delta'_j)$.

Define $\mathbf{S} = \mathbf{S}_n : C_n(X) \to C_n(X)$ recursively on n as follows:

$\mathbf{S}_{-1} = \text{id}$ and $\mathbf{S}_0 = \text{id}$.

For $n \geq 1$, let $\sigma : \Delta \to X$ be a singular n-simplex, where $\Delta = [w_0, \ldots, w_n]$. Define $\mathbf{S}_n(\sigma) = \mathbf{b}_\Delta(\mathbf{S}_{n-1}(\partial\sigma))$. Given $\mathbf{x} = \sum_j m_j \sigma_j \in C_n(X)$, define $\mathbf{S}_n(\mathbf{x}) = \sum_j m_j \mathbf{S}_n(\sigma_j)$.



Claim 7.1. For any $\mathbf{x} \in C_n(X)$, $\quad \mathbf{S}\partial\mathbf{x} = \partial\mathbf{S}\mathbf{x}$.

Define another homomorphism $\mathbf{T}_n : C_n(X) \to C_{n+1}(X)$ by recursion on n as follows:

$\mathbf{T}_{-1} = 0$ and $\mathbf{T}_0 = 0$.

For $n \geq 1$, let $\sigma : \Delta \to X$ be a singular n-simplex. Define $\mathbf{T}_n(\sigma) = \mathbf{b}_\Delta(\sigma - \mathbf{T}_{n-1}(\partial\sigma))$. Given $\mathbf{x} = \sum_j m_j \sigma_j \in C_n(X)$, define $\mathbf{T}_n(\mathbf{x}) = \sum_j m_j \mathbf{T}_n(\sigma_j)$.

Claim 7.2. For any $\mathbf{x} \in C_n(X)$, $\quad \mathbf{T}\partial\mathbf{x} + \partial\mathbf{T}\mathbf{x} = \mathbf{x} - \mathbf{S}\mathbf{x}$.

Claim 7.3. If $\mathbf{y} \in C_n(X)$ is subordinate to $\mathcal{U}$, then so are $\mathbf{S}\mathbf{y}$ and $\mathbf{T}\mathbf{y}$.

The desired chain $\mathbf{x}'$ is

$$\mathbf{x}' = \mathbf{T}\partial\mathbf{x} + \mathbf{T}\partial\mathbf{S}\mathbf{x} + \mathbf{T}\partial\mathbf{S}\mathbf{S}\mathbf{x} + \cdots + \mathbf{T}\partial \overbrace{\mathbf{S}\cdots\mathbf{S}}^{k-1 \text{ times}} \mathbf{x} + \overbrace{\mathbf{S}\cdots\mathbf{S}}^{k \text{ times}}\mathbf{x}$$

for some $k \in \mathbb{N}$.

It follows from Claims 7.1 and 7.3 and the assumption that $\partial\mathbf{x}$ is subordinate to $\mathcal{U}$ that each term $\mathbf{T}\partial\mathbf{S}\cdots\mathbf{S}\mathbf{x}$ is subordinate to $\mathcal{U}$. For the last term, each time we apply $\mathbf{S}$, the diameters of the domains of the singular n-simplices in the result goes down by a factor of $\frac{n}{n+1}$ (not proved here), hence can be made arbitrarily small by iteration of $\mathbf{S}$. In particular, the diameters of these domains can be made smaller than the Lebesgue number for the covers $\{\sigma^{-1}(U) : U \in \mathcal{U}\}$ of Δ^n, for each σ from $\mathbf{x}$. Thus for large enough k, $\overbrace{\mathbf{S}\cdots\mathbf{S}}^{k \text{ times}}\mathbf{x}$ will be subordinate to $\mathcal{U}$.

The desired chain $\mathbf{z}$ is

$$\mathbf{z} = \mathbf{T}\mathbf{x} + \mathbf{T}\mathbf{S}\mathbf{x} + \mathbf{T}\mathbf{S}\mathbf{S}\mathbf{x} + \cdots + \mathbf{T} \overbrace{\mathbf{S}\cdots\mathbf{S}}^{k-1 \text{ times}} \mathbf{x}.$$

The equality $\mathbf{x} - \mathbf{x}' = \partial\mathbf{z}$ follows from Claim 7.2 (applied $k-1$ times). □

Exact sequences

Exact sequence: a sequence of (abelian) groups A_n and homomorphisms ϕ_n

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\phi_{n+1}} A_n \xrightarrow{\phi_n} A_{n-1} \xrightarrow{\phi_{n-1}} \cdots$$

satisfying $\mathrm{Im}(\phi_{n+1}) = \mathrm{Ker}(\phi_n)$ for all n.

Example.

- $A \xrightarrow{a} B \xrightarrow{0} C$ exact means a is onto.
- $A \xrightarrow{0} B \xrightarrow{b} C$ exact means b is one-to-one.
- $A \xrightarrow{0} B \xrightarrow{b} C \xrightarrow{0} D$ exact means b is an isomorphism, so $B \cong C$.
- $A \xrightarrow{0} B \xrightarrow{b} C \xrightarrow{c} D \xrightarrow{0} E$ exact implies b is one-to-one (so $\mathrm{Im}(b) \cong B$) and $D \cong {}^{C}/_{\mathrm{Im}(b)}$.

Note that if $\mathbf{f} : \widetilde{H}_m(X) \to \widetilde{H}_n(Y)$ is a homomorphism, then

$$\begin{aligned} \mathbf{x} \in \mathrm{Ker}(\mathbf{f}) &\quad \text{if and only if} \quad \mathbf{f}(\mathbf{x}) \equiv 0 \text{ in } \widetilde{H}_n(Y) \\ \mathbf{y} \in \mathrm{Im}(\mathbf{f}) &\quad \text{if and only if} \quad \text{there exists } \mathbf{x} \in \widetilde{H}_m(X) \text{ such that } \mathbf{f}(\mathbf{x}) \equiv \mathbf{y} \text{ in } \widetilde{H}_n(Y) \end{aligned}$$

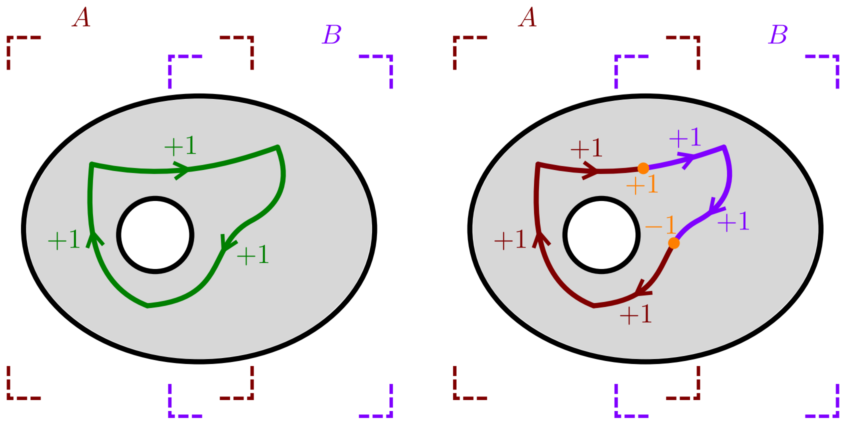
Mayer-Vietoris Theorem. Suppose A, B are subsets of X such that $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$. The sequence

$$\cdots \longrightarrow \widetilde{H}_n(A \cap B) \xrightarrow{\mathbf{r}_*} \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \xrightarrow{\mathbf{s}_*} \widetilde{H}_n(X) \xrightarrow{\mathbf{d}} \widetilde{H}_{n-1}(A \cap B) \longrightarrow \cdots \longrightarrow \widetilde{H}_0(X) \longrightarrow 0$$

where

$$\begin{aligned} \mathbf{r}_*(\mathbf{x}) &= (\mathbf{x}, -\mathbf{x}) \\ \mathbf{s}_*(\mathbf{a}, \mathbf{b}) &= \mathbf{a} + \mathbf{b} \\ \mathbf{d}(\mathbf{x}) &= \partial\mathbf{a} = -\partial\mathbf{b}, \text{ where } \mathbf{a} \in C_n(\mathrm{Int}(A)),\ \mathbf{b} \in C_n(\mathrm{Int}(B)), \text{ and } \mathbf{x} \equiv \mathbf{a} + \mathbf{b} \text{ in } \widetilde{H}_n(X), \end{aligned}$$

is exact at each place to the left of $\widetilde{H}_0(X)$, and if $A \cap B \neq \emptyset$ it is exact at $\widetilde{H}_0(X)$ too.



Proof. (1) $\mathbf{r}_*$ is well defined: exercise.

(2) $\mathbf{s}_*$ is well defined: exercise.

(3) $\mathbf{d}$ is well defined: Suppose $\mathbf{x} \equiv \mathbf{a}_1 + \mathbf{b}_1 \equiv \mathbf{a}_2 + \mathbf{b}_2$ in $\widetilde{H}_n(X)$ for some $\mathbf{a}_1, \mathbf{a}_2 \in C_n(\mathrm{Int}(A))$ and $\mathbf{b}_1, \mathbf{b}_2 \in C_n(\mathrm{Int}(B))$. So $\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{b}_1 - \mathbf{b}_2 = \partial\mathbf{z}$ for some $\mathbf{z} \in C_{n+1}(X)$. Express $\mathbf{z} = \mathbf{a}_3 + \mathbf{b}_3 + \partial\mathbf{z}'$ for some $\mathbf{a}_3 \in C_{n+1}(\mathrm{Int}(A))$, $\mathbf{b}_3 \in C_{n+1}(\mathrm{Int}(B))$, and $\mathbf{z}' \in C_{n+2}(X)$. Then $\partial\mathbf{z} = \partial\mathbf{a}_3 + \partial\mathbf{b}_3$, and so $\mathbf{a}_1 - \mathbf{a}_2 - \partial\mathbf{a}_3 = -(\mathbf{b}_1 - \mathbf{b}_2 - \partial\mathbf{b}_3)$, which means this chain belongs to $C_n(A \cap B)$. Now $\partial\mathbf{a}_1 \equiv \partial\mathbf{a}_2$ in $\widetilde{H}_{n-1}(A \cap B)$ because $\partial\mathbf{a}_1 - \partial\mathbf{a}_2 = \partial(\mathbf{a}_1 - \mathbf{a}_2 - \partial\mathbf{a}_3)$.

(4) $\mathrm{Im}(\mathbf{r}_*) \subseteq \mathrm{Ker}(\mathbf{s}_*)$: exercise.

(5) $\mathrm{Ker}(\mathbf{s}_*) \subseteq \mathrm{Im}(\mathbf{r}_*)$: Suppose $\mathbf{a}+\mathbf{b} \equiv 0$ in $\widetilde{H}_n(X)$, so $\mathbf{a}+\mathbf{b} = \partial\mathbf{z}$ for some $\mathbf{z} \in C_{n+1}(X)$. Express $\mathbf{z} = \mathbf{a}_1+\mathbf{b}_1+\partial\mathbf{z}'$ for some $\mathbf{a}_1 \in C_{n+1}(\mathrm{Int}(A))$, $\mathbf{b}_1 \in C_{n+1}(\mathrm{Int}(B))$, and $\mathbf{z}' \in C_{n+2}(X)$. So $\mathbf{a} - \partial\mathbf{a}_1 = -(\mathbf{b} - \partial\mathbf{b}_1)$, which means this chain belongs to $C_n(A \cap B)$. Note this chain is a cycle, and its image under $\mathbf{r}_*$ is $(\mathbf{a} - \partial\mathbf{a}_1, \mathbf{b} - \partial\mathbf{b}_1)$, which equals ($\equiv$) $(\mathbf{a}, \mathbf{b})$ in $\widetilde{H}_n(A) \oplus \widetilde{H}_n(B)$.

(6) $\mathrm{Im}(\mathbf{s}_*) \subseteq \mathrm{Ker}(\mathbf{d})$: exercise.

(7) $\mathrm{Ker}(\mathbf{d}) \subseteq \mathrm{Im}(\mathbf{s}_*)$: exercise.

(8) $\mathrm{Im}(\mathbf{d}) \subseteq \mathrm{Ker}(\mathbf{r}_*)$: exercise.

(9) $\mathrm{Ker}(\mathbf{r}_*) \subseteq \mathrm{Im}(\mathbf{d})$: exercise.

(10) If $A \cap B \neq \emptyset$, then $\mathbf{s}_* : \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \to \widetilde{H}_0(X)$ is onto: Given $\mathbf{x} \in \widetilde{H}_0(X)$, write $\mathbf{x} = \sum_j m_j a_j + \sum_k o_k b_k$ where $a_j \in A$ and $b_k \in B$ (viewed as 0-simplices). So $\sum_j m_j + \sum_k o_k = 0$. Let $p \in A \cap B$. Define $\mathbf{a} = \sum_j m_j a_j + \sum_k o_k p$ and $\mathbf{b} = \sum_j m_j p + \sum_k o_k b_k$. Then $(\mathbf{a}, \mathbf{b}) \in \widetilde{H}_0(A) \oplus \widetilde{H}_0(B)$, and $\mathbf{s}_*(\mathbf{a}, \mathbf{b}) \equiv \mathbf{x}$ in $\widetilde{H}_0(X)$.

□

Example. $X = \mathbb{S}^1$, A, B arcs ($\approx \mathbb{D}^1$), $A \cap B \simeq \mathbb{S}^0$.

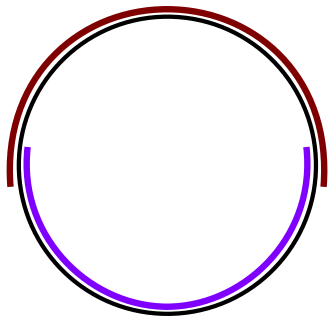
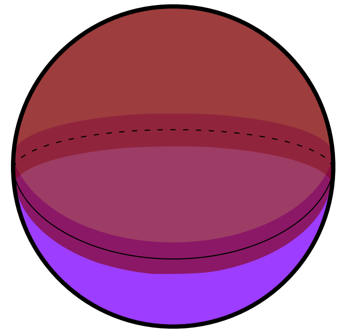
$$\begin{array}{cccccccccccccccccc}
\cdots \longrightarrow & \widetilde{H}_2(A) \oplus \widetilde{H}_2(B) & \xrightarrow{\mathbf{s}_*} & \widetilde{H}_2(X) & \xrightarrow{\mathbf{d}} & \widetilde{H}_1(A \cap B) & \xrightarrow{\mathbf{r}_*} & \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) & \xrightarrow{\mathbf{s}_*} & \widetilde{H}_1(X) & \xrightarrow{\mathbf{d}} & \widetilde{H}_0(A \cap B) & \xrightarrow{\mathbf{r}_*} & \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) & \xrightarrow{\mathbf{s}_*} & \widetilde{H}_0(X) & \longrightarrow 0 \\
\cdots \longrightarrow & 0 & \longrightarrow & \widetilde{H}_2(\mathbb{S}^1) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \widetilde{H}_1(\mathbb{S}^1) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \widetilde{H}_0(\mathbb{S}^1) & \longrightarrow 0
\end{array}$$

So $\widetilde{H}_n(\mathbb{S}^1) \cong \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$

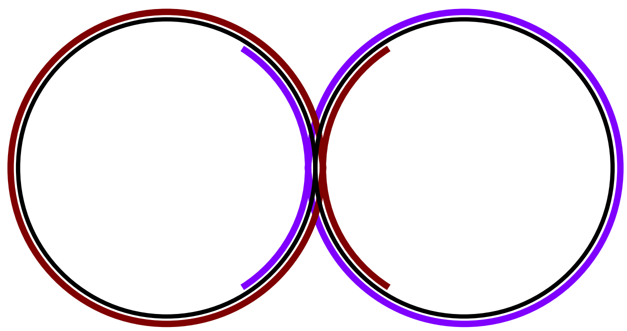
Generator of $\widetilde{H}_1(\mathbb{S}^1)$: $\mathbf{s}^1 = \sigma^1_+ - \sigma^1_-$, where σ^1_+ is the top half of the circle traced from right to left, and σ^1_- is the reflection of σ^1_+ in line $\{\vec{x} \in \mathbb{R}^2 : x_2 = 0\}$. Note that $\partial\sigma^1_+ = \partial\sigma^1_- = \mathbf{s}^0$.

Example. Generally, for $m \geq 0$: $\widetilde{H}_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z} & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$

$\widetilde{H}_m(\mathbb{S}^m)$ has a generator of the form $\mathbf{s}^m = \mathbf{s}^m_+ - \mathbf{s}^m_-$, where $\mathbf{s}^m_-$ is reflection of $\mathbf{s}^m_+$ in hyperplane $\{\vec{x} \in \mathbb{R}^{m+1} : x_{m+1} = 0\}$. Moreover, $\partial\mathbf{s}^m_+ = \partial\mathbf{s}^m_- = \mathbf{s}^{m-1}$.

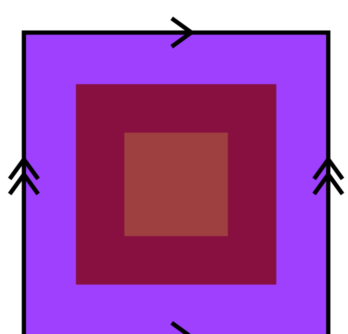
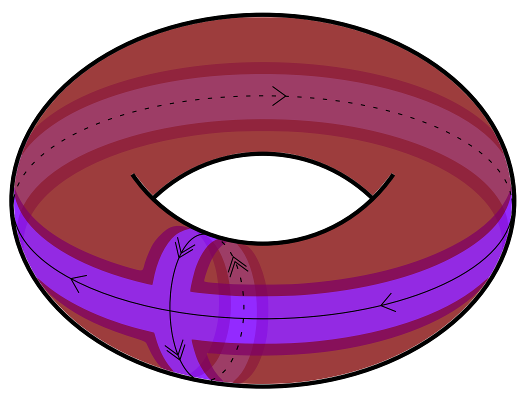
Example. $X = \mathbb{S}^1 \vee \mathbb{S}^1$, $A, B \simeq \mathbb{S}^1$, $A \cap B$ contractible ($\simeq$ single point space $*$).



$$\begin{array}{ccccccccccccccccc}
\cdots \longrightarrow \widetilde{H}_2(A) \oplus \widetilde{H}_2(B) & \xrightarrow{\mathbf{s}_*} & \widetilde{H}_2(X) & \xrightarrow{\mathbf{d}} & \widetilde{H}_1(A\cap B) & \xrightarrow{\mathbf{r}_*} & \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) & \xrightarrow{\mathbf{s}_*} & \widetilde{H}_1(X) & \xrightarrow{\mathbf{d}} & \widetilde{H}_0(A\cap B) & \xrightarrow{\mathbf{r}_*} & \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) & \xrightarrow{\mathbf{s}_*} & \widetilde{H}_0(X) & \longrightarrow 0 \\
\cdots \longrightarrow \quad 0 & \longrightarrow & \widetilde{H}_2(\mathbb{S}^1 \vee \mathbb{S}^1) & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \widetilde{H}_1(\mathbb{S}^1 \vee \mathbb{S}^1) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \widetilde{H}_0(\mathbb{S}^1 \vee \mathbb{S}^1) & \longrightarrow 0
\end{array}$$

So $\widetilde{H}_n(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$

Example. $X = \mathbb{T}$ torus, $A = \mathbb{D}^2$, $B \simeq \mathbb{S}^1 \vee \mathbb{S}^1$, $A \cap B \simeq \mathbb{S}^1$.

$$\begin{array}{ccccccccccccccccc}
\cdots \longrightarrow \widetilde{H}_2(A) \oplus \widetilde{H}_2(B) & \xrightarrow{\mathbf{s}_*} & \widetilde{H}_2(X) & \xrightarrow{\mathbf{d}} & \widetilde{H}_1(A\cap B) & \xrightarrow{\mathbf{r}_*} & \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) & \xrightarrow{\mathbf{s}_*} & \widetilde{H}_1(X) & \xrightarrow{\mathbf{d}} & \widetilde{H}_0(A\cap B) & \xrightarrow{\mathbf{r}_*} & \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) & \xrightarrow{\mathbf{s}_*} & \widetilde{H}_0(X) & \longrightarrow 0 \\
\cdots \longrightarrow \quad 0 & \longrightarrow & \widetilde{H}_2(\mathbb{T}) & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \widetilde{H}_1(\mathbb{T}) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \widetilde{H}_0(\mathbb{T}) & \longrightarrow 0
\end{array}$$

Can show $\mathbf{r}_* : \widetilde{H}_1(A \cap B) \to \widetilde{H}_1(A) \oplus \widetilde{H}_1(B)$ is 0.

So $\widetilde{H}_n(\mathbb{T}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases}$

Example. $K =$ Klein bottle. $\widetilde{H}_n(K) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$

Example. $\mathbb{R}\mathrm{P}^2 =$ Real projective plane. $\widetilde{H}_n(\mathbb{R}\mathrm{P}^2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$

Example. Let $m \geq 0$. If m is even, $\widetilde{H}_n(\mathbb{R}\mathrm{P}^m) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is odd and } 0 < n < m \\ 0 & \text{otherwise.} \end{cases}$

If m is odd, $\widetilde{H}_n(\mathbb{R}\mathrm{P}^m) \cong \begin{cases} \mathbb{Z} & \text{if } n = m \text{ (odd)} \\ \mathbb{Z}_2 & \text{if } n \text{ is odd and } 0 < n < m \\ 0 & \text{otherwise.} \end{cases}$

1.5 (optional) Long exact sequence for pair

This is an exact sequence relating the homology groups of X, A, and ${}^{X}\!/_{A}$, when A is a nice subspace of X. It is among the Eilenberg-Steenrod axioms for homology theory, and is common to see in treatments of homology. It could be used instead of the Mayer-Vietoris sequence for some calculations, but we won't use it in these notes.

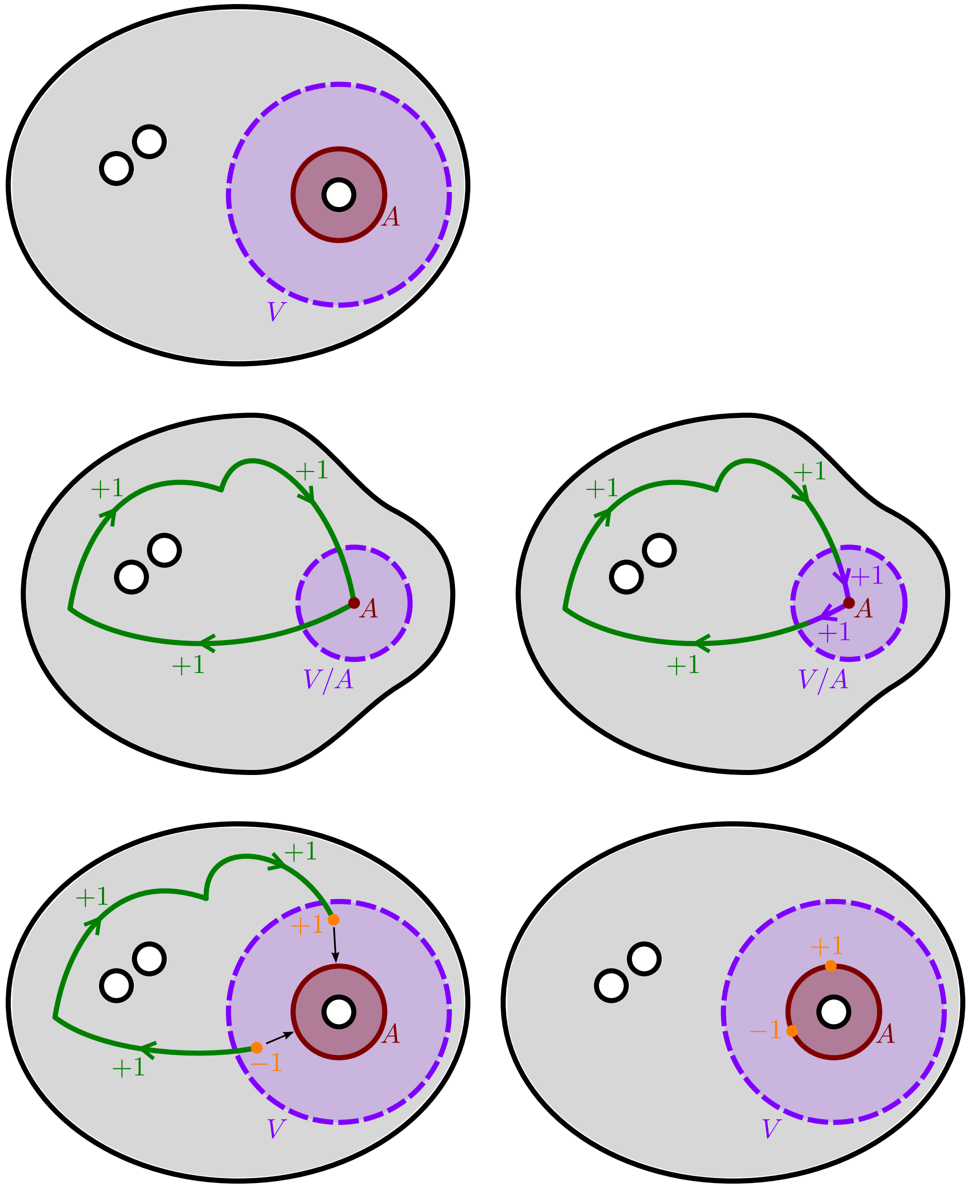
Theorem 8. Let A be a non-empty closed subset of X, and suppose there is a neighborhood V of A in X and a retraction $r : V \to A$ which is homotopic (in V) to id_V (so A is a deformation retract of V). The sequence

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{q_*} \widetilde{H}_n\left({}^{X}\!/_{A}\right) \xrightarrow{\mathbf{d}_A} \widetilde{H}_{n-1}(A) \longrightarrow \cdots \longrightarrow \widetilde{H}_0\left({}^{X}\!/_{A}\right) \longrightarrow 0$$

where

$$i_*(\mathbf{x}) = \mathbf{x} \qquad \text{(homomorphism induced by the inclusion } i : A \to X)$$
$$q_*(\mathbf{x}) = \mathbf{x} \qquad \text{(homomorphism induced by the quotient projection } q : X \to {}^{X}\!/_{A})$$
$$\mathbf{d}_A(\mathbf{x}) = r_*(\partial \mathbf{u}) = -r_*(\partial \mathbf{v}), \text{ where } \mathbf{u} \in C_n(X \setminus A),\ \mathbf{v} \in C_n\left({}^{V}\!/_{A}\right), \text{ and } \mathbf{x} \equiv \mathbf{u} + \mathbf{v} \text{ in } \widetilde{H}_n\left({}^{X}\!/_{A}\right),$$

is exact.



Proof. (1) $\mathbf{d}_A$ is well defined: Suppose $\mathbf{x} \equiv \mathbf{u}_1 + \mathbf{v}_1 \equiv \mathbf{u}_2 + \mathbf{v}_2$ in $\widetilde{H}_n\left({}^X/_A\right)$ for some $\mathbf{u}_1, \mathbf{u}_2 \in C_n(X \smallsetminus A)$ and $\mathbf{v}_1, \mathbf{v}_2 \in C_n\left({}^V/_A\right)$. So $\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{v}_1 - \mathbf{v}_2 = \partial \mathbf{z}$ for some $\mathbf{z} \in C_{n+1}\left({}^X/_A\right)$. Express $\mathbf{z} = \mathbf{u}_3 + \mathbf{v}_3 + \partial \mathbf{z}'$ for some $\mathbf{u}_3 \in C_{n+1}(X \smallsetminus A)$, $\mathbf{v}_3 \in C_{n+1}\left({}^V/_A\right)$, and $\mathbf{z}' \in C_{n+2}\left({}^X/_A\right)$. Then $\partial \mathbf{z} = \partial \mathbf{u}_3 + \partial \mathbf{v}_3$, and so $\mathbf{u}_1 - \mathbf{u}_2 - \partial \mathbf{u}_3 = -(\mathbf{v}_1 - \mathbf{v}_2 - \partial \mathbf{v}_3)$, which means this chain belongs to $C_n(V \smallsetminus A)$. Now $r_*(\partial \mathbf{u}_1) \equiv r_*(\partial \mathbf{u}_2)$ in $\widetilde{H}_{n-1}(A)$ because $r_*(\partial \mathbf{u}_1) - r_*(\partial \mathbf{u}_2) = \partial r_*(\mathbf{u}_1 - \mathbf{u}_2 - \partial \mathbf{u}_3)$.

(2) $\mathrm{Im}(i_*) \subseteq \mathrm{Ker}(q_*)$: exercise.

(3) $\mathrm{Ker}(q_*) \subseteq \mathrm{Im}(i_*)$: exercise.

(4) $\mathrm{Im}(q_*) \subseteq \mathrm{Ker}(\mathbf{d}_A)$: exercise.

(5) $\mathrm{Ker}(\mathbf{d}_A) \subseteq \mathrm{Im}(q_*)$: exercise.

(6) $\mathrm{Im}(\mathbf{d}_A) \subseteq \mathrm{Ker}(i_*)$: exercise.

(7) $\mathrm{Ker}(i_*) \subseteq \mathrm{Im}(\mathbf{d}_A)$: exercise.

□

1.6 (optional) Algebraic abstractions

There are lots of exact sequences in homology theory. There is an algebraic explanation for where these come from, which we state below without proof. We will not use these concepts further in these notes.

Chain complex: a sequence of (abelian) groups C_n and homomorphisms ∂_n

$$\cdots \longrightarrow C_{n+1} \stackrel{\partial_{n+1}}{\longrightarrow} C_n \stackrel{\partial_n}{\longrightarrow} C_{n-1} \stackrel{\partial_{n-1}}{\longrightarrow} \cdots$$

satisfying $\mathrm{Im}(\partial_{n+1}) \subseteq \mathrm{Ker}(\partial_n)$ for all n.

Homology groups: $H_n = {}^{\mathrm{Ker}(\partial_n)}/_{\mathrm{Im}(\partial_{n+1})}$

Let C_n, ∂_n and C'_n, ∂'_n be two chain complexes, with homology groups H_n and H'_n.
A **chain map** between the chain complexes C_n, ∂_n and C'_n, ∂'_n is a sequence of homomorphisms $\mathbf{f}_n : C_n \to C'_n$ satisfying $\mathbf{f}_{n-1} \circ \partial_n = \partial'_n \circ \mathbf{f}_n$ for each n.

On all these functions we often omit subscripts, so this becomes $\mathbf{f}\partial = \partial'\mathbf{f}$.

Proposition 9. A chain map $\mathbf{f}$ induces a sequence of homomorphisms $\mathbf{f}_* : H_n \to H'_n$ on homology groups.

Example. If $f : X \to Y$ is a continuous function, then $f_\sharp : C_n(X) \to C_n(Y)$ is a chain map (cf. Claim 4.1).

A **chain homotopy** between chain maps $\mathbf{f}$ and $\mathbf{g}$ is a sequence of homomorphisms $\mathbf{h} : C_n \to C'_{n+1}$ satisfying $\partial'\mathbf{h} - \mathbf{h}\partial = \mathbf{g} - \mathbf{f}$. We say $\mathbf{f}$ and $\mathbf{g}$ are **chain homotopic**.

Example. The function $\mathbf{P}$ in the proof of Theorem 6 is a chain homotopy between the chain maps $f_\sharp : C_n(X) \to C_n(Y)$ and $g_\sharp : C_n(X) \to C_n(Y)$ (notation mismatched a little here: there we had $f : X \to Y$, $f_\sharp : C_n(X) \to C_n(Y)$, and $f_* : H_n(X) \to H_n(Y)$; here we have $\mathbf{f} : C_n \to C'_n$ and $\mathbf{f}_* : H_n \to H'_n$).

Example. The function $\mathbf{T}$ in the sketch of the proof of the Subdivision Lemma is a chain homotopy between the chain maps $\mathbf{S} : C_n(X) \to C_n(X)$ and $\mathrm{id} : C_n(X) \to C_n(X)$.

Proposition 10. If $\mathbf{f}$ and $\mathbf{g}$ are chain homotopic chain maps, then $\mathbf{f}_* = \mathbf{g}_*$.

Theorem 11. Let C_n, ∂_n, C'_n, ∂'_n, and C''_n, ∂''_n be chain complexes, and let $\mathbf{i}$ and $\mathbf{j}$ be chain maps where $\mathbf{i}_n : C_n \to C'_n$ and $\mathbf{j}_n : C'_n \to C''_n$. Suppose that for each n the sequence

$$0 \longrightarrow C_n \stackrel{\mathbf{i}}{\longrightarrow} C'_n \stackrel{\mathbf{j}}{\longrightarrow} C''_n \longrightarrow 0$$

is exact. Then there exists a sequence of homomorphisms $\mathbf{d}_n : H''_n \to H_{n-1}$ such that

$$\cdots \longrightarrow H_n \stackrel{\mathbf{i}_*}{\longrightarrow} H'_n \stackrel{\mathbf{j}_*}{\longrightarrow} H''_n \stackrel{\mathbf{d}}{\longrightarrow} H_{n-1} \stackrel{\mathbf{i}_*}{\longrightarrow} H'_{n-1} \longrightarrow \cdots$$

is exact.

Example. The function $\mathbf{d}$ from the Mayer-Vietoris Theorem satisfies the conclusion of Theorem 11 for the exact sequence

$$0 \longrightarrow C_n(A \cap B) \stackrel{\mathbf{r}}{\longrightarrow} C_n(A) \oplus C_n(B) \stackrel{\mathbf{s}}{\longrightarrow} C_n^{\mathcal{U}}(X) \longrightarrow 0$$

where $\mathbf{r}(\mathbf{x}) = (\mathbf{x}, -\mathbf{x})$ and $\mathbf{s}(\mathbf{a}, \mathbf{b}) = \mathbf{a} + \mathbf{b}$, and $C_n^{\mathcal{U}}(X)$ denotes the set of all n-chains in X which are subordinate to the cover $\mathcal{U} = \{A, B\}$. One must also confirm that the homology groupds of the chain complex $C_n^{\mathcal{U}}(X)$ are isomorphic to $H_n(X)$, which follows from the Subdivision Lemma.

Example. The function $\mathbf{d}_A$ from the long exact sequence for the pair X, A satisfies the conclusion of Theorem 11 for the exact sequence

$$0 \longrightarrow C_n(A) \stackrel{i_\sharp}{\longrightarrow} C_n(X) \stackrel{\mathbf{q}}{\longrightarrow} {}^{C_n(X)}/_{C_n(A)} \longrightarrow 0$$

where $i_\sharp$ is the homomorphism induced by the inclusion map $i : A \to X$ and $\mathbf{q}$ is the quotient homomorphism. One must also confirm that the homology groups for the chain complex ${}^{C_n(X)}/_{C_n(A)}$ are isomorphic to $H_n\left({}^{X}/_{A}\right)$, which follows from the so-called **Excision Theorem**.

1.7 (optional) Unreduced homology with coefficients

Let G be an arbitrary abelian group. An alternative version of homology is made by allowing coefficients in chains to be from G instead of $\mathbb{Z}$. We'll also change our convention for -1-chains to get "unreduced" homology. We will only consider this version of homology in this section and in the optional section below on the Borsuk-Ulam Theorem, and there we will only work with $G = \mathbb{Z}_2$.

The definition is exactly the same except for these points:

An ***n*-chain** in X is a (finite) (formal) sum of singular n-simplices $\mathbf{x} = \sum_j m_j \sigma_j$, where $m_j \in G$. $C_n(X;G) = \{n\text{-chains in } X\}$.

$C_n(X;G) = 0$ for all $n \leq -1$.

If $\mathbf{x}$ is a 0-chain, then $\partial \mathbf{x} = 0$.

(Unreduced) Homology group with *G* coefficients $H_n(X;G) = {}^{\mathrm{Ker}(\partial_n)}/_{\mathrm{Im}(\partial_{n+1})}$.

Without re-doing the proofs (they hardly need changing), we collect the relevant results below.

Theorem 2; *G*. If X has path components X_α, $\alpha \in I$, then $H_n(X;G) \cong \bigoplus_{\alpha \in I} H_n(X_\alpha;G)$ for all n.

Note: In unreduced homology, this result holds for $n = 0$.

Theorem 3; *G*. If X has k path components, then $H_0(X;G) \cong G \oplus G \oplus \cdots \oplus G$ (k times). Specifically, if $p_1, \ldots, p_k \in X$ are any points from each path component, then $H_0(X;G) = \langle p_1, \ldots, p_k \rangle$ (viewing p_i here as singular 0-simplices).

Given a continuous function $f : X \to Y$, define the **induced homomorphism** $f_* : H_n(X;G) \to H_n(Y;G)$ by $f_*(\sum_j m_j \sigma_j) = \sum_j m_j \cdot (f \circ \sigma_j)$.

Theorem 5; *G*.

- $\mathrm{id}_* = \mathrm{id}$
- $(f \circ g)_* = f_* \circ g_*$ (whenever $g : W \to X$ and $f : X \to Y$).

Theorem 6; *G*. If $f \simeq g$, then $f_* = g_*$.

Corollary 7; *G*. If $X \simeq Y$, then $H_n(X;G) \cong H_n(Y;G)$ for all n.

Mayer-Vietoris Theorem; *G*. Suppose A, B are subsets of X such that $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$. The sequence

$$\cdots \longrightarrow H_n(A \cap B;G) \xrightarrow{\mathbf{r}_*} H_n(A;G) \oplus H_n(B;G) \xrightarrow{\mathbf{s}_*} H_n(X;G) \xrightarrow{\mathbf{d}} H_{n-1}(A \cap B;G) \longrightarrow \cdots \longrightarrow H_0(X;G) \longrightarrow 0$$

where

$$\begin{aligned} \mathbf{r}_*(\mathbf{x}) &= (\mathbf{x}, -\mathbf{x}) \\ \mathbf{s}_*(\mathbf{a}, \mathbf{b}) &= \mathbf{a} + \mathbf{b} \\ \mathbf{d}(\mathbf{x}) &= \partial \mathbf{a} = -\partial \mathbf{b}, \text{ where } \mathbf{a} \in C_n(\mathrm{Int}(A);G),\ \mathbf{b} \in C_n(\mathrm{Int}(B);G), \text{ and } \mathbf{x} \equiv \mathbf{a} + \mathbf{b} \text{ in } H_n(X;G), \end{aligned}$$

is exact.

Note: In unreduced homology, we don't need the assumption that $A \cap B \neq \emptyset$ to get exactness at $H_0(X;G)$.

Example. Let $*$ be the space with just one point. Then $H_n(*;G) \cong \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$

Example. $H_n(\mathbb{S}^0;G) \cong \begin{cases} G \oplus G & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$

Example. For $m \geq 1$, $H_n(\mathbb{S}^m;G) \cong \begin{cases} G & \text{if } n = 0 \text{ or } m \\ 0 & \text{otherwise.} \end{cases}$

Recall the groups $\widetilde{H}_n(\mathbb{RP}^m)$. The result for $H_n(\mathbb{RP}^m;\mathbb{Z}_2)$ is much tidier.

Example. For $m \geq 1$, $H_n(\mathbb{RP}^m;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq n \leq m \\ 0 & \text{otherwise.} \end{cases}$

Let $p : \mathbb{S}^m \to \mathbb{RP}^m$ denote the quotient projection. Then $H_m(\mathbb{RP}^m;\mathbb{Z}_2)$ is generated (over $\mathbb{Z}_2$) by $p_*(\mathbf{s}^m_+)$. Note $p_*(\mathbf{s}^m_+) \equiv p_*(\mathbf{s}^m_-)$ in $H_m(\mathbb{RP}^m;\mathbb{Z}_2)$.

1.8 Exercises

Exercise 1. Consider $X = \mathbb{R}^2$. Identify Δ^1 with $[0,1]$, and define singular 1-simplices $\sigma_1, \sigma_2 : \Delta^1 \to \mathbb{R}^2$ by $\sigma_1(t) = (\cos(\pi t), \sin(\pi t))$ and $\sigma_2(t) = (\cos(\pi(t+1)), \sin(\pi(t+1)))$, so that the 1-chain $\mathbf{x} = \sigma_1 + \sigma_2$ corresponds to the unit circle. Find a 2-chain $\mathbf{y}$ in $\mathbb{R}^2$ such that $\partial\mathbf{y} = \mathbf{x}$.

Exercise 2. Let σ be a singular 1-simplex in X. Let $\sigma^{\leftarrow}$ denote the singular 1-simplex which is identical with σ except that it parameterizes the path in the opposite direction; i.e. $\sigma^{\leftarrow}(t) = \sigma(1-t)$ (identifying Δ^1 with $[0,1]$).

Observe that in $C_1(X)$, $\sigma^{\leftarrow} \neq -\sigma$. However, show that there is a 2-chain $\mathbf{y}$ such that $\sigma^{\leftarrow} + \sigma = \partial\mathbf{y}$ (i.e. $\sigma^{\leftarrow} + \sigma \equiv 0$ in $\widetilde{H}_1(X)$).

Exercise 3. Let $\sigma : \Delta^1 \to \mathbb{S}^1$ be a path which goes once around $\mathbb{S}^1$ counter-clockwise starting at the point $(1,0) \in \mathbb{S}^1$. Prove that $\sigma \equiv \mathbf{s}^1$ in $\widetilde{H}_1(\mathbb{S}^1)$.

Exercise 4. Let $\sigma_1 : \Delta^1 \to \mathbb{S}^1$ be a path which goes once around $\mathbb{S}^1$ counter-clockwise starting at some point $\vec{x}_1 \in \mathbb{S}^1$, and let $\sigma_2 : \Delta^1 \to \mathbb{S}^1$ be another path which goes once around $\mathbb{S}^1$ counter-clockwise starting at a different point $\vec{x}_2 \in \mathbb{S}^1$. Prove that $\sigma_1 \equiv \sigma_2$ in $\widetilde{H}_1(\mathbb{S}^1)$.

Exercise 5. Let $\sigma_1 : \Delta^1 \to \mathbb{S}^1$ be a path which goes once around $\mathbb{S}^1$ counter-clockwise starting at the point $(1,0) \in \mathbb{S}^1$, and let $\sigma_2 : \Delta^1 \to \mathbb{S}^1$ be another path which goes n times around $\mathbb{S}^1$ counter-clockwise starting at the point $(1,0) \in \mathbb{S}^1$. Prove that $n \cdot \sigma_1 \equiv \sigma_2$ in $\widetilde{H}_1(\mathbb{S}^1)$.

Exercise 6. Confirm (directly from the definition of $\widetilde{H}_n$) that $\widetilde{H}_n(*) = 0$ for all n, where $*$ is the space with just one point.

Exercise 7. Fill in missing details in the proof of the Subdivision Lemma.

Exercise 8. Fill in missing parts of the proof of the Mayer-Vietoris Theorem.

Exercise 9. Compute the groups $\widetilde{H}_n(\mathbb{S}^m)$ by induction on m.

Exercise 10. Compute the reduced homology groups of the wedge of m circles.

Exercise 11. Suppose $X = \mathrm{Int}(X_1) \cup \mathrm{Int}(X_2)$, and that $X_1 \cap X_2$ is contractible. Prove that $\widetilde{H}_n(X) \cong \widetilde{H}_n(X_1) \oplus \widetilde{H}_n(X_2)$ for all n.

Exercise 12. Compute the reduced homology groups of the torus, the Klein bottle, the real projective plane.

Exercise 13. Compute the reduced homology groups of the Warsaw circle.

Exercise 14. Compute the groups $\widetilde{H}_n(\mathbb{RP}^m)$.

Exercise 15. Let $c : X \to Y$ be a constant map. Prove that $c_* : \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ is 0 for all n.

Exercise 16. Let X be a space such that $\widetilde{H}_n(X) \neq 0$ for some n. Observe that id_X is not homotopic to a constant map.

Exercise 17. Fill in missing parts of the proof of the long exact sequence for the pair.

Exercise 18. Use the long exact sequence for the pair $\mathbb{D}^m$, $\mathbb{S}^{m-1} = \mathrm{Bd}(\mathbb{D}^m)$ to compute the reduced homology groups of $\mathbb{S}^m = {}^{\mathbb{D}^m}/_{\mathbb{S}^{m-1}}$, by induction on m.

Exercise 19. Compute the groups $H_n(\mathbb{RP}^m; \mathbb{Z}_2)$.

2 Applications

2.1 Brouwer's fixed point theorem

Prerequisite homology content: $\widetilde{H}_m(\mathbb{S}^m)$, $\widetilde{H}_m(\mathbb{D}^m)$; induced homomorphisms on homology.

A **retraction** of X onto a subset $A \subset X$ is a continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$.

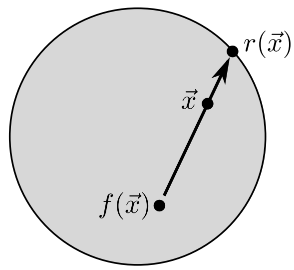
Lemma 1. Let $m \geq 1$. There is no retraction from $\mathbb{D}^m$ to $\mathrm{Bd}(\mathbb{D}^m) = \mathbb{S}^{m-1}$.

Proof. Suppose $r : \mathbb{D}^m \to \mathbb{S}^{m-1}$ is a retraction. Let $i : \mathbb{S}^{m-1} \to \mathbb{D}^m$ be the inclusion map (identity).

$r \circ i = \mathrm{id}_{\mathbb{S}^{m-1}}$, so $r_* \circ i_* = \mathrm{id}$ on $\widetilde{H}_{m-1}(\mathbb{S}^{m-1}) \cong \mathbb{Z}$. But $\widetilde{H}_{m-1}(\mathbb{D}^m) = 0$, so $i_* = r_* = 0$. This is a contradiction. □

Brouwer's Fixed Point Theorem. Let $m \geq 1$, and let $f : \mathbb{D}^m \to \mathbb{D}^m$ be any continuous function. There exists a point $\vec{x} \in \mathbb{D}^n$ such that $f(\vec{x}) = \vec{x}$.

Proof. Suppose $f : \mathbb{D}^m \to \mathbb{D}^m$ has no fixed point. Define $r : \mathbb{D}^m \to \mathbb{S}^{m-1}$ by $r(\vec{x})$ = the point where the ray from $f(\vec{x})$ to $\vec{x}$ meets $\mathbb{S}^{m-1}$. This is a retraction, which contradicts the previous lemma. □



2.2 Embeddings of spheres

Prerequisite homology content: $\widetilde{H}_n(\mathbb{S}^m)$, $\widetilde{H}_n(\mathbb{D}^m)$; Mayer-Vietoris Theorem.

Let $k, m \geq 0$.

Lemma 2. Suppose $h : \mathbb{D}^k \to \mathbb{S}^m$ is an embedding. Then $\widetilde{H}_n(\mathbb{S}^m \setminus h(\mathbb{D}^k)) = 0$ for all n.

It is tempting to guess that since $\mathbb{D}^k \simeq *$, we should have $\mathbb{S}^m \setminus h(\mathbb{D}^k) \simeq \mathbb{S}^m \setminus *$ (from which this lemma would follow immediately since $\mathbb{S}^m \setminus * \approx \mathbb{R}^m \simeq *$). The Alexander Horned Sphere is an embedding $h : \mathbb{D}^3 \to \mathbb{S}^3$ such that $\mathbb{S}^3 \setminus h(\mathbb{D}^3) \not\simeq \mathbb{S}^3 \setminus *$.

Proof. Identify $\mathbb{D}^k$ with $[0,1]^k$.

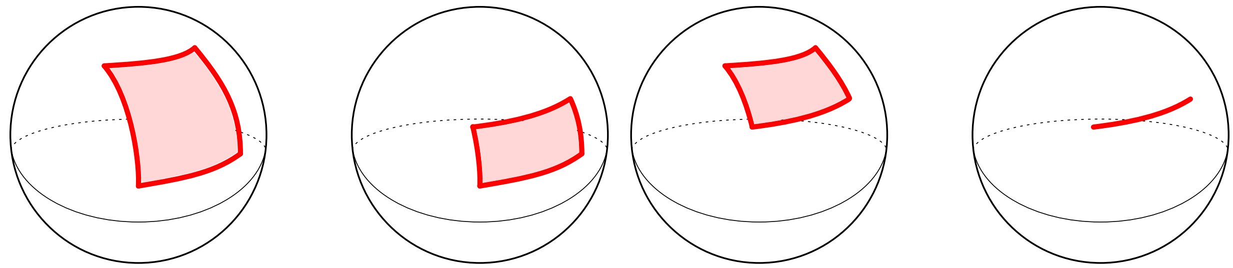
We proceed by induction on k. Case $k = 0$ is trivial, since $\mathbb{S}^m \setminus h(\mathbb{D}^0) \approx \mathbb{R}^m$. Assume the claim holds for k. Fix $n \geq 0$.

Given a closed interval I, split it in half: $I = I_- \cup I_+$, $I_- \cap I_+ = \{\text{midpoint of } I\}$.

Claim 2.1. Let $\mathbf{y}$ be an n-cycle in $\mathbb{S}^m \setminus h([0,1]^k \times I)$. If $\mathbf{y} \not\equiv 0$ in $\widetilde{H}_n(\mathbb{S}^m \setminus h([0,1]^k \times I))$, then $\mathbf{y} \not\equiv 0$ in $\widetilde{H}_n(\mathbb{S}^m \setminus h([0,1]^k \times I_-))$ or $\mathbf{y} \not\equiv 0$ in $\widetilde{H}_n(\mathbb{S}^m \setminus h([0,1]^k \times I_+))$.

Proof of Claim 2.1. Consider

$$Z = \mathbb{S}^m \setminus h([0,1]^k \times I)$$
$$A = \mathbb{S}^m \setminus h([0,1]^k \times I_-)$$
$$B = \mathbb{S}^m \setminus h([0,1]^k \times I_+).$$



Note A and B are open, $A \cap B = Z$, and $A \cup B = \mathbb{S}^m \smallsetminus h([0,1]^k \times \{\text{midpoint of } I\})$. By induction, $\widetilde{H}_n(A \cup B) = \widetilde{H}_{n+1}(A \cup B) = 0$. According to the Mayer-Vietoris sequence, this implies the function $\mathbf{x} \mapsto (\mathbf{x}, -\mathbf{x})$ is an isomorphism from $\widetilde{H}_n(Z)$ to $\widetilde{H}_n(A) \oplus \widetilde{H}_n(B)$. So $(\mathbf{y}, -\mathbf{y}) \not\equiv 0$ in $\widetilde{H}_n(\mathbb{S}^m \smallsetminus h([0,1]^k \times I_-)) \oplus \widetilde{H}_n(\mathbb{S}^m \smallsetminus h([0,1]^k \times I_+))$, which means either $\mathbf{y} \not\equiv 0$ in $\widetilde{H}_n(\mathbb{S}^m \smallsetminus h([0,1]^k \times I_-))$ or $\mathbf{y} \not\equiv 0$ in $\widetilde{H}_n(\mathbb{S}^m \smallsetminus h([0,1]^k \times I_+))$. □(Claim 2.1)

Suppose there exists a non-zero $\mathbf{y} \in \widetilde{H}_n(\mathbb{S}^m \smallsetminus h([0,1]^{k+1}))$. Apply Claim 2.1 repeatedly to obtain nested sequence of closed intervals

$$[0,1] = I_0 \supset I_1 \supset I_2 \supset \cdots$$

with $\operatorname{diam}(I_j) \to 0$ such that $\mathbf{y} \not\equiv 0$ in $\widetilde{H}_n(\mathbb{S}^m \smallsetminus h([0,1]^k \times I_j))$ for each j.

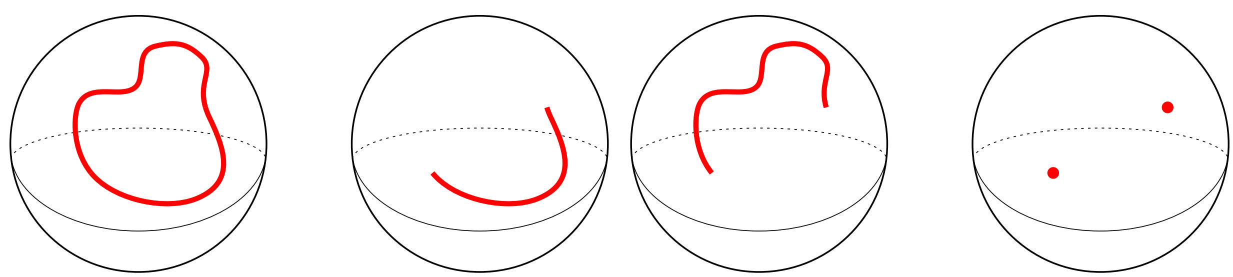
The intersection $\bigcap_{j=0}^{\infty} I_j$ is a single point $\{t\}$. By induction, $\mathbf{y} \equiv 0$ in $\widetilde{H}_n(\mathbb{S}^m \smallsetminus h([0,1]^k \times \{t\}))$, so $\mathbf{y} = \partial \mathbf{z}$ for some $\mathbf{z} \in C_{n+1}(\mathbb{S}^m \smallsetminus h([0,1]^k \times \{t\}))$. Since $\mathbb{S}^m \smallsetminus h([0,1]^k \times I_j)$ are increasing open sets whose union is $\mathbb{S}^m \smallsetminus h([0,1]^k \times \{t\})$, by compactness we have $\mathbf{z} \in C_{n+1}(\mathbb{S}^m \smallsetminus h([0,1]^k \times I_j))$ for some j. But this means $\mathbf{y} \equiv 0$ in $\widetilde{H}_n(\mathbb{S}^m \smallsetminus h([0,1]^k \times I_j))$, a contradiction. □

Lemma 3. Suppose $h : \mathbb{S}^k \to \mathbb{S}^m$ is an embedding which is not onto. Then $\widetilde{H}_n(\mathbb{S}^m \smallsetminus h(\mathbb{S}^k)) \cong \begin{cases} \mathbb{Z} & \text{if } n = m-k-1 \\ 0 & \text{otherwise.} \end{cases}$

Proof. We proceed by induction on k. For $k = 0$: $\mathbb{S}^m \smallsetminus h(\mathbb{S}^0) \approx \mathbb{S}^{m-1} \times \mathbb{R} \simeq \mathbb{S}^{m-1}$, so $\widetilde{H}_n(\mathbb{S}^m \smallsetminus h(\mathbb{S}^0)) \cong \begin{cases} \mathbb{Z} & \text{if } n = m-1 \\ 0 & \text{otherwise.} \end{cases}$

Suppose the claim holds for k. Let $\mathbb{S}^{k+1}_- = \{(x_1, \ldots, x_{k+2}) \in \mathbb{S}^{k+1} : x_{k+2} \le 0\}$ and $\mathbb{S}^{k+1}_+ = \{(x_1, \ldots, x_{k+2}) \in \mathbb{S}^{k+1} : x_{k+2} \ge 0\}$. Note $\mathbb{S}^{k+1}_- \approx \mathbb{S}^{k+1}_+ \approx \mathbb{D}^{k+1}$, and $\mathbb{S}^{k+1}_- \cap \mathbb{S}^{k+1}_+ = \mathbb{S}^k$. Consider

$$\begin{aligned} Z &= \mathbb{S}^m \smallsetminus h(\mathbb{S}^{k+1}) \\ A &= \mathbb{S}^m \smallsetminus h(\mathbb{S}^{k+1}_-) \\ B &= \mathbb{S}^m \smallsetminus h(\mathbb{S}^{k+1}_+). \end{aligned}$$



Note A and B are open, $A \cap B = Z$, and $A \cup B = \mathbb{S}^m \smallsetminus h(\mathbb{S}^k)$. By Lemma 2, $\widetilde{H}_n(A) \oplus \widetilde{H}_n(B) = 0$. According to the Mayer-Vietoris sequence, this implies that $\widetilde{H}_n(\mathbb{S}^m \smallsetminus h(\mathbb{S}^k)) \cong \widetilde{H}_{n-1}(Z)$. The claim now follows by induction. □

Corollary 4. Any embedding of $\mathbb{S}^m$ into $\mathbb{S}^m$ must be onto.

Corollary 5. If $k > m$, then there is no one-to-one continuous function from an open set $U \subset \mathbb{S}^k$ to $\mathbb{S}^m$.

Corollary 6. Jordan Curve Theorem: If $h : \mathbb{S}^{m-1} \to \mathbb{S}^m$ is an embedding, then $\mathbb{S}^m \smallsetminus h(\mathbb{S}^{m-1})$ has two components.

Proof. By Theorem 3, $\widetilde{H}_0(\mathbb{S}^m \setminus h(\mathbb{S}^{m-1})) \cong \mathbb{Z}$, so $\mathbb{S}^m \setminus h(\mathbb{S}^{m-1})$ has two path components. Since $\mathbb{S}^m \setminus h(\mathbb{S}^{m-1})$ is an open subset of $\mathbb{S}^m$, its components coincide with its path components. □

Theorem 7. Invariance of Domain: If $U \subseteq \mathbb{S}^m$ is open in $\mathbb{S}^m$ and $h : U \to \mathbb{S}^m$ is a one-to-one continuous function, then $h(U)$ is open in $\mathbb{S}^m$.

Proof. Let $\vec{x} \in U$, and let $D \subset U$ be a closed m-ball ($D \approx \mathbb{D}^m$) centered at $\vec{x}$. By compactness, h is an embedding of $\partial D \approx \mathbb{S}^{m-1}$ into $\mathbb{S}^m$. By the Jordan Curve Theorem, $\mathbb{S}^m \setminus h(\partial D)$ has two path components, which are evidently $h(D \setminus \partial D)$ and $\mathbb{S}^m \setminus h(D)$ (these are both path connected, and their union is $\mathbb{S}^m \setminus h(\partial D)$). Since $\mathbb{S}^m \setminus h(\partial D)$ is an open subset of $\mathbb{S}^m$, its components coincide with its path components. Hence $\mathbb{S}^m \setminus h(D)$ is closed in $\mathbb{S}^m \setminus h(\partial D)$, which means $h(D \setminus \partial D)$ is open in $\mathbb{S}^m \setminus h(\partial D)$, and so is an open neighborhood of $h(\vec{x})$ in $\mathbb{S}^m$. □

Corollary 8. Let M be a compact m-dimensional manifold and let N be a connected m-dimensional (Hausdorff) manifold. If $h : M \to N$ is an embedding, then h is onto.

Proof. $h(M)$ is closed in N since M is compact and N is Hausdorff. That $h(M)$ is open in N follows from Invariance of Domain. □

2.3 Degree

Prerequisite homology content: $\widetilde{H}_m(\mathbb{S}^m)$; induced homomorphisms on homology.

Let $m \geq 1$, and let $f : \mathbb{S}^m \to \mathbb{S}^m$ be a continuous function.

Definition. Let $\mathbf{x}$ be a generator for $\widetilde{H}_m(\mathbb{S}^m) \cong \mathbb{Z}$ (so $\mathbf{x} \equiv \mathbf{s}_+^m - \mathbf{s}_-^m$ in $\widetilde{H}_m(\mathbb{S}^m)$ or $\mathbf{x} \equiv \mathbf{s}_-^m - \mathbf{s}_+^m$ in $\widetilde{H}_m(\mathbb{S}^m)$). The induced homomorphism $f_* : \widetilde{H}_m(\mathbb{S}^m) \to \widetilde{H}_m(\mathbb{S}^m)$ takes $\mathbf{x}$ to $k \cdot \mathbf{x}$ for some $k \in \mathbb{Z}$. This k is the **degree** of f.

Proposition 9.
- $\deg(\mathrm{id}) = 1$.
- $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
- If $f \simeq g$, then $\deg(f) = \deg(g)$.

Proposition 10. If $h : \mathbb{S}^m \to \mathbb{S}^m$ is a homeomorphism, then $\deg(h) = \pm 1$, and $\deg(h^{-1}) = \deg(h)$.

Corollary 11. If $h : \mathbb{S}^m \to \mathbb{S}^m$ is a homeomorphism, then $\deg(h^{-1} \circ f \circ h) = \deg(f)$.

Lemma 12. If f is the reflection in a hyperplane through the origin, then $\deg(f) = -1$

Proof. By previous result, we can reduce to the case where f is the reflection in the hyperplane $\{\vec{x} \in \mathbb{R}^{m+1} : x_{m+1} = 0\}$.

Recall that $\widetilde{H}_m(\mathbb{S}^m)$ has a generator of the form $\mathbf{s}_+^m - \mathbf{s}_-^m$, where $\mathbf{s}_-^m$ is reflection of $\mathbf{s}_+^m$ in hyperplane $\{\vec{x} \in \mathbb{R}^{m+1} : x_{m+1} = 0\}$. Clearly the reflection in the hyperplane $\{\vec{x} \in \mathbb{R}^{m+1} : x_{m+1} = 0\}$ negates this generator. □

Lemma 13. The antipodal map $a : \mathbb{S}^m \to \mathbb{S}^m$ defined by $a(\vec{x}) = -\vec{x}$ has degree $(-1)^{m+1}$.

Proof. a is the composition of $m + 1$ reflections in hyperplanes through the origin. □

Hairy Ball Theorem. If m is even, then $\mathbb{S}^m$ has no continuous field of non-zero tangent vectors.

Proof. Suppose for a contradiction that $F : \mathbb{S}^m \to \mathbb{R}^{m+1}$ is a continuous vector field such that for all $\vec{x} \in \mathbb{S}^m$, $F(\vec{x}) \neq 0$, and $F(\vec{x})$ is orthogonal to $\vec{x}$. By normalizing, we may assume $\|F(\vec{x})\| = 1$ for all $\vec{x} \in \mathbb{S}^m$.

Define $h : \mathbb{S}^m \times [0, 1] \to \mathbb{S}^m$ by $h(\vec{x}, t) = \cos(\pi t) \cdot \vec{x} + \sin(\pi t) \cdot F(\vec{x})$. Then h is a homotopy from $\mathrm{id}_{\mathbb{S}^m}$ to the antipodal map a on $\mathbb{S}^m$. But $\deg(\mathrm{id}_{\mathbb{S}^m}) = 1$ and $\deg(a) = (-1)^{m+1} = -1$ since m is even. This is a contradiction. □

If m is odd, then $\mathbb{S}^m$ does have a continuous field of tangent vectors (see Exercise 6).

For the next result we identify $\mathbb{R}^2$ with $\mathbb{C}$, so $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Lemma 14. Let $n \geq 1$. The angle n-tupling map $\mathbb{S}^1 \to \mathbb{S}^1$ defined by $z \mapsto z^n$ has degree n.

Fundamental Theorem of Algebra. Let $p(z)$ be a polynomial of degree $n \geq 1$ with complex coefficients. Then $p(z)$ has a root in $\mathbb{C}$.

Proof. Write $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. We may assume $a_n = 1$. Write $e(z) = a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, so $p(z) = z^n + e(z)$.

Let $R \geq 1$ be large enough so that $\frac{|e(z)|}{|z^n|} < 1$ for all z with $|z| = R$. Let $S = \{z \in \mathbb{C} : |z| = R\} \approx \mathbb{S}^1$. By definition of R, $p(z)$ has no root on S, so we consider $p|_S$ as a function $S \to \mathbb{C} \setminus \{0\}$. The function $F : S \times [0,1] \to \mathbb{C} \setminus \{0\}$ defined by $F(z,t) = z^n + te(z)$ is a homotopy between $z \mapsto z^n$ and $p|_S$. It follows from Lemma 14 that $p|_S$ is not homotopic (in $\mathbb{C} \setminus \{0\}$) to a constant function.

On the other hand, if p has no root in $D = \{z \in \mathbb{C} : |z| \leq R\}$, then $p|_D : D \to \mathbb{C} \setminus \{0\}$ is a continuous extension of $p|_S$ to the disk D, which means $p|_S$ is homotopic (in $\mathbb{C} \setminus \{0\}$) to a constant function, a contradiction. □

Definition. Let $f : \mathbb{S}^1 \to \mathbb{R}^2$ be a continuous function and let $\vec{p} \in \mathbb{R}^2$ such that $\vec{p} \notin f(\mathbb{S}^1)$. The **winding number** of f with respect to $\vec{p}$ is the degree of the function $w_{\vec{p}} : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $w_{\vec{p}}(\vec{x}) = \frac{f(\vec{x}) - \vec{p}}{\|f(\vec{x}) - \vec{p}\|}$.

Proposition 15. Let $f, g : \mathbb{S}^1 \to \mathbb{R}^2$, and suppose there is a homotopy F from f to g such that $\vec{p} \notin F(\mathbb{S}^1 \times [0,1])$. Then the winding number of f with respect to $\vec{p}$ equals the winding number of g with respect to $\vec{p}$.

2.4 (optional) Borsuk-Ulam Theorem

Prerequisite homology content: $H_m(\mathbb{S}^m; \mathbb{Z}_2)$, $H_n(\mathbb{RP}^m; \mathbb{Z}_2)$; induced homomorphisms on homology; degree.

We will work with unreduced homology with $\mathbb{Z}_2$ coefficients (see Section 1.7). Keep in mind that if $G \cong H \cong \mathbb{Z}_2$, then there are only two homomorphisms from G to H: the 0 function, and the isomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$.

Theorem 16. If $f : \mathbb{S}^m \to \mathbb{S}^m$ is an odd function (i.e. if $f(-\vec{x}) = -f(\vec{x})$ for all $\vec{x} \in \mathbb{S}^m$), then $\deg(f)$ is odd.

Proof.

Claim 16.1. If $\deg(f)$ is even, then $f_* : H_m(\mathbb{S}^m; \mathbb{Z}_2) \to H_m(\mathbb{S}^m; \mathbb{Z}_2)$ is 0.

Thus it suffices to prove that $f_* : H_m(\mathbb{S}^m; \mathbb{Z}_2) \to H_m(\mathbb{S}^m; \mathbb{Z}_2)$ is not 0 (i.e. is the isomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$).

Since f is odd, it induces a function $\overline{f} : \mathbb{RP}^m \to \mathbb{RP}^m$. Let $p : \mathbb{S}^m \to \mathbb{RP}^m$ denote the quotient projection. Then $\overline{f}p = pf$. Given an n-simplex σ in $\mathbb{RP}^m$, let $\widehat{\sigma}_a$ and $\widehat{\sigma}_b$ be the two lifts of σ in $\mathbb{S}^m$ under p (i.e. $\sigma = p \circ \widehat{\sigma}_a = p \circ \widehat{\sigma}_b$, and $\widehat{\sigma}_b = -\widehat{\sigma}_a$). Define $\boldsymbol{\tau}(\sigma) = \widehat{\sigma}_a + \widehat{\sigma}_b$. Given an n-chain $\mathbf{x} = \sum_j m_j \sigma_j$ in $\mathbb{RP}^m$ (with $\mathbb{Z}_2$ coefficients), define $\boldsymbol{\tau}(\mathbf{x}) = \sum_j m_j \boldsymbol{\tau}(\sigma_j)$; this formula defines a function $C_n(\mathbb{RP}^m; \mathbb{Z}_2) \to C_n(\mathbb{S}^m; \mathbb{Z}_2)$.

Claim 16.2. $\boldsymbol{\tau}$ induces a function $\boldsymbol{\tau}_* : H_n(\mathbb{RP}^m; \mathbb{Z}_2) \to H_n(\mathbb{S}^m; \mathbb{Z}_2)$, and $f_* \boldsymbol{\tau}_* = \boldsymbol{\tau}_* \overline{f}_*$.

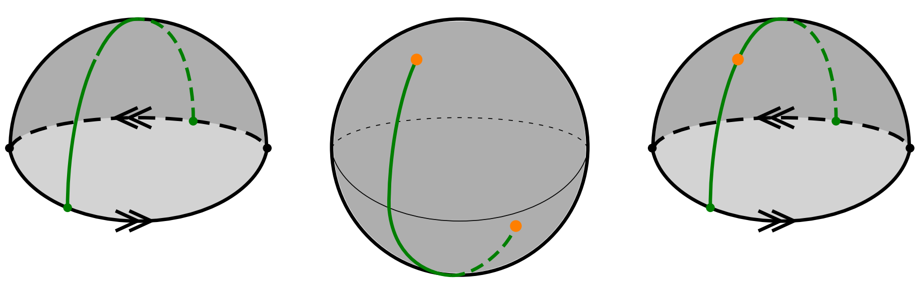
[*Hint*: It will help to establish these two facts:

- The function $\boldsymbol{\tau} : C_n(\mathbb{RP}^m; \mathbb{Z}_2) \to C_n(\mathbb{S}^m; \mathbb{Z}_2)$ is one-to-one;
- For the functions $\boldsymbol{\tau} : C_n(\mathbb{RP}^m; \mathbb{Z}_2) \to C_n(\mathbb{S}^m; \mathbb{Z}_2)$ and $p_\sharp : C_n(\mathbb{S}^m; \mathbb{Z}_2) \to C_n(\mathbb{RP}^m; \mathbb{Z}_2)$, $\mathrm{Im}(\boldsymbol{\tau}) = \mathrm{Ker}(p_\sharp)$.]

Claim 16.3. $\boldsymbol{\tau}_* : H_m(\mathbb{RP}^m; \mathbb{Z}_2) \to H_m(\mathbb{S}^m; \mathbb{Z}_2)$ is the isomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$.

In light of the Claims 16.2 and 16.3, to prove $f_* : H_m(\mathbb{S}^m; \mathbb{Z}_2) \to H_m(\mathbb{S}^m; \mathbb{Z}_2)$ is the isomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$, it suffices to prove that $\overline{f}_* : H_m(\mathbb{RP}^m; \mathbb{Z}_2) \to H_m(\mathbb{RP}^m; \mathbb{Z}_2)$ is the isomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$.

Define $\mathbf{d} : H_n(\mathbb{RP}^m; \mathbb{Z}_2) \to H_{n-1}(\mathbb{RP}^m; \mathbb{Z}_2)$ as follows: given $\mathbf{x} \in H_n(\mathbb{RP}^m; \mathbb{Z}_2)$, choose an n-chain $\widehat{\mathbf{x}} \in C_n(\mathbb{S}^m; \mathbb{Z}_2)$ such that $p_\sharp(\widehat{\mathbf{x}}) = \mathbf{x}$, and let $\mathbf{d}(\mathbf{x})$ be the unique element $\mathbf{y} \in H_{n-1}(\mathbb{RP}^m; \mathbb{Z}_2)$ such that $\boldsymbol{\tau}_*(\mathbf{y}) \equiv \partial \widehat{\mathbf{x}}$ in $H_{n-1}(\mathbb{S}^m; \mathbb{Z}_2)$.



Claim 16.4. This function $\mathbf{d}$ is well-defined. That is:

- $p_\sharp : C_n(\mathbb{S}^m;\mathbb{Z}_2) \to C_n(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2)$ is onto;
- Given an n-chain $\widehat{\mathbf{x}} \in C_n(\mathbb{S}^m;\mathbb{Z}_2)$ such that $p_*(\widehat{\mathbf{x}}) = \mathbf{x}$, there exists an element $\mathbf{y} \in H_{n-1}(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2)$ such that $\boldsymbol{\tau}(\mathbf{y}) = \partial\widehat{\mathbf{x}}$;
- Given another n-chain $\widehat{\mathbf{x}}' \in C_n(\mathbb{S}^m;\mathbb{Z}_2)$ such that $p_*(\widehat{\mathbf{x}})$ is an n-cycle and $p_*(\widehat{\mathbf{x}}) \equiv \mathbf{x}$ in $H_n(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2)$, and given an element $\mathbf{y}' \in H_{n-1}(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2)$ such that $\boldsymbol{\tau}(\mathbf{y}') = \partial\widehat{\mathbf{x}}'$, we have $\mathbf{y} \equiv \mathbf{y}'$ in $H_{n-1}(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2)$.

Claim 16.5. For $1 \le n \le m$, the function $\mathbf{d} : H_n(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) \to H_{n-1}(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2)$ is the isomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$.

Claim 16.6. $\overline{f}_*\mathbf{d} = \mathbf{d}\overline{f}_*$.

$$\begin{array}{ccccccccc}
H_m(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) & \xrightarrow{\mathbf{d}} & H_{m-1}(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) & \xrightarrow{\mathbf{d}} & \cdots & \xrightarrow{\mathbf{d}} & H_1(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) & \xrightarrow{\mathbf{d}} & H_0(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) \\
\downarrow\overline{f}_* & & \downarrow\overline{f}_* & & \cdots & & \downarrow\overline{f}_* & & \downarrow\overline{f}_* \\
H_m(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) & \xrightarrow{\mathbf{d}} & H_{m-1}(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) & \xrightarrow{\mathbf{d}} & \cdots & \xrightarrow{\mathbf{d}} & H_1(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) & \xrightarrow{\mathbf{d}} & H_0(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2)
\end{array}$$

Clearly $\overline{f}_* : H_0(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) \to H_0(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2)$ is the isomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$. From Claims 16.5 and 16.6, it then follows by induction on n that $\overline{f}_* : H_n(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) \to H_n(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2)$ is the isomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$ for each $0 \le n \le m$. In particular, $\overline{f}_* : H_m(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) \to H_m(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2)$ is the isomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$. □

Borsuk-Ulam Theorem. Let $g : \mathbb{S}^m \to \mathbb{R}^m$ be a continuous function. There exists a point $\vec{x} \in \mathbb{S}^m$ such that $g(\vec{x}) = g(-\vec{x})$.

Proof. Suppose for a contradiction that $g(\vec{x}) \ne g(-\vec{x})$ for all $\vec{x} \in \mathbb{S}^m$. Define $f : \mathbb{S}^m \to \mathbb{S}^{m-1}$ by $f(\vec{x}) = \frac{g(\vec{x})-g(-\vec{x})}{\|g(\vec{x})-g(-\vec{x})\|}$. Observe that f is odd.

The restriction of f to the equator $\mathbb{S}^{m-1} \subset \mathbb{S}^m$ is an odd function from $\mathbb{S}^{m-1}$ to $\mathbb{S}^{m-1}$, hence has odd degree by Theorem 16. On the other hand, this restriction is homotopic (in $\mathbb{S}^{m-1}$) to a constant map (using the upper or lower hemisphere of $\mathbb{S}^m$), hence has degree 0. This is a contradiction. □

2.5 Covering dimension

Prerequisite homology content: $\widetilde{H}_m(\mathbb{S}^m)$; induced homomorphisms on homology.

Recall that if $\{U_1,\dots,U_k\}$ is a cover of a space X, a **partition of unity** subordinate to $\{U_1,\dots,U_k\}$ is a set of continuous functions $\{\phi_j : j = 1,\dots,k\}$ from X to $[0,1]$ such that $\phi_j|_{X\smallsetminus U_j} = 0$, and $\sum_{j=1}^k \phi_j(x) = 1$ for each $x \in X$. We will use the fact that any open cover of a compact space admits a partition of unity.

The following result represents the challenging part of the proof that the covering dimension of $\mathbb{R}^m$ is m.

Theorem 17. For any cover of $\mathbb{D}^m$ by sufficiently small open sets, there is a point of $\mathbb{D}^m$ which belongs to at least $m+1$ cover elements.

Proof. Identify $\mathbb{D}^m$ with $\Delta^m = [v_0,\dots,v_m]$. Let $\{U_1,\dots,U_k\}$ be an open cover of Δ^m such that no U_j meets all of the $m+1$ faces (of dimension $m-1$) of Δ^m. Suppose for a contradiction that no point of Δ^m belongs to $m+1$ of these sets U_j.

For each $j = 1,\dots,k$ choose $i(j) \in \{0,\dots,m\}$ such that $v_{i(j)}$ belongs to each face of Δ^m that U_j intersects (or choose $i(j) \in \{0,\dots,m\}$ arbitrarily if U_j does not meet any face).

Let $\{\phi_j : j = 1,\dots,k\}$ be a partition of unity for Δ^m subordinate to the cover $\{U_1,\dots,U_k\}$. Define $f : \Delta^m \to \mathrm{Bd}(\Delta^m)$ by $f(\vec{x}) = \sum_{j=1}^k \phi_j(\vec{x})v_{i(j)}$.

Claim 17.1. The range of f really is contained in $\mathrm{Bd}(\Delta^m)$.

Claim 17.2. For each face $\Delta' = [v_0,\dots,\widehat{v_i},\dots,v_m]$ of Δ^m, $\quad f(\Delta') \subseteq \Delta'$.

The restriction $f|_{\mathrm{Bd}(\Delta^m)} : \mathrm{Bd}(\Delta^m) \to \mathrm{Bd}(\Delta^m)$ is homotopic (in $\mathrm{Bd}(\Delta^m)$) to a constant map, since $\mathrm{Bd}(\Delta^m) \approx \mathbb{S}^{m-1}$ and $f|_{\mathrm{Bd}(\Delta^m)}$ extends over $\Delta^m \approx \mathbb{D}^m$.

On the other hand, we may use a straight-line homotopy on each face Δ' individually from $f|_{\Delta'}$ to $\mathrm{id}_{\Delta'}$, to obtain a homotopy between $f|_{\mathrm{Bd}(\Delta^m)}$ and $\mathrm{id}_{\mathrm{Bd}(\Delta^m)}$, a contradiction. (Note it follows from Claim 17.2 that the intersection of two faces of Δ^m is invariant under f, hence the straight-line homotopies coincide on these intersections.) □

2.6 Exercises

Exercise 1. Prove that there is no retraction from the Möbius strip to its boundary circle.

Exercise 2. Prove that if $f : \mathbb{S}^m \to \mathbb{S}^m$ is homotopic to a constant function, then there exist points $\vec{x}, \vec{y} \in \mathbb{S}^m$ such that $f(\vec{x}) = -\vec{x}$ and $f(\vec{y}) = -\vec{y}$.

Exercise 3. Come up with a degree 2 map $\mathbb{S}^2 \to \mathbb{S}^2$.

Exercise 4. Let A be an $m \times m$ matrix with positive real entries. Prove that A has a positive eigenvalue, with a corresponding eigenvector having positive real entries.

[*Hint*: Let $B = \{(x_1, \ldots, x_m) \in \mathbb{S}^{m-1} : x_i \geq 0 \text{ for each } i\}$, and define a function f on B by $f(\vec{x}) = \frac{A\vec{x}}{\|A\vec{x}\|}$.]

Exercise 5. Consider the m-dimensional cube $[-1,1]^m$. Suppose $B_1, \ldots, B_m$ are closed subsets of $[-1,1]^m$ such that B_i separates $\{\vec{x} \in [-1,1]^m : x_i = -1\}$ from $\{\vec{x} \in [-1,1]^m : x_i = 1\}$ for each i. Prove that $\bigcap_{i=1}^m B_i \neq \emptyset$.

[*Hint*: For each i let U_i^- and U_i^+ be open sets such that $U_i^- \cup U_i^+ = [-1,1]^m \setminus B_i$, $\{\vec{x} \in [-1,1]^m : x_i = -1\} \subset U_i^-$, and $\{\vec{x} \in [-1,1]^m : x_i = 1\} \subset U_i^+$. Define a function f on $[-1,1]^m$ by $f(\vec{x}) = (f_1(\vec{x}), \ldots, f_m(\vec{x}))$, where

$$f_i(\vec{x}) = \begin{cases} x_i + d(x_i, B_i) & \text{if } x_i \in U_i^- \\ x_i - d(x_i, B_i) & \text{if } x_i \in U_i^+ \\ x_i & \text{if } x_i \in B_i. \end{cases}$$

Exercise 6. Show that if m is odd, then $\mathbb{S}^m$ admits a continuous field of tangent vectors.

Exercise 7. Prove that if m is even and $f : \mathbb{S}^m \to \mathbb{S}^m$ is continuous, then there is a point $\vec{x} \in \mathbb{S}^m$ such that $f(\vec{x}) = \pm\vec{x}$.

Exercise 8. Prove Lemma 14. (cf. Exercises 3 and 5 from Chapter 1.)

Exercise 9. Prove Proposition 15.

Exercise 10. Fill in the proofs of the claims in Theorem 16.

Exercise 11. Prove the **Ham Sandwich Theorem:** Given any m Jordan measurable regions $A_1, \ldots, A_m$ in $\mathbb{R}^m$, there is a single $(m-1)$-dimensional hyperplane which splits each A_i in half.

[*Hint*: Identify $\mathbb{R}^m$ with $\mathbb{R}^m \times \{1\} \subset \mathbb{R}^{m+1}$. Given $\vec{x} \in \mathbb{S}^m$, let $P(\vec{x})$ be the hyperplane in $\mathbb{R}^{m+1}$ through the origin orthogonal to $\vec{x}$. Define $g : \mathbb{S}^m \to \mathbb{R}^m$ by $g(\vec{x}) = (g_1(\vec{x}), \ldots, g_m(\vec{x}))$, where $g_i(\vec{x})$ is the volume of the part of A_i on the same side of $P(\vec{x})$ as $\vec{x}$.]

Exercise 12. Let $F_1, \ldots, F_{m+1} \subseteq \mathbb{S}^m$ be closed sets which cover $\mathbb{S}^m$. Prove that there exists $\vec{x} \in \mathbb{S}^m$ and $i \in \{1, \ldots, m+1\}$ such that both $\vec{x}$ and $-\vec{x}$ belong to F_i.

[*Hint*: Define $g : \mathbb{S}^m \to \mathbb{R}^m$ by $g(\vec{x}) = (g_1(\vec{x}), \ldots, g_m(\vec{x}))$, where $g_i(\vec{x}) = d(\vec{x}, F_i)$ (the distance from $\vec{x}$ to F_i).]