Paths in Euclidean space

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Nipissing University

July 23, 2014
Summer Topology Conference
College of Staten Island

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- γ_i is constant on each fiber $m_i^{-1}(s)$, $s \in [0,1]$, for both $i = 1, 2$; and
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Metric d on Π:

$$d([\gamma_1],[\gamma_2]) = \inf\{d_{\sup}(\lambda_1, \lambda_2) : \lambda_1 \in [\gamma_1], \lambda_2 \in [\gamma_2]\}$$

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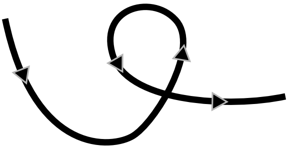

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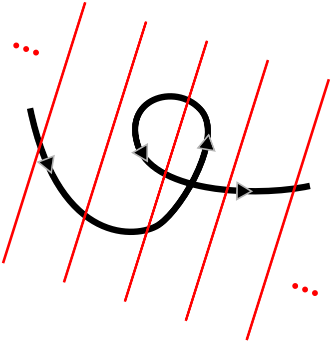

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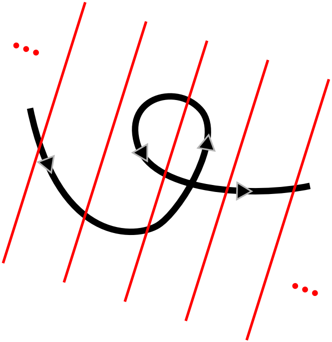

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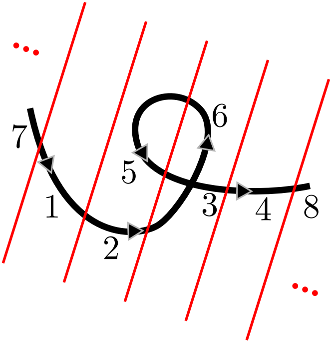

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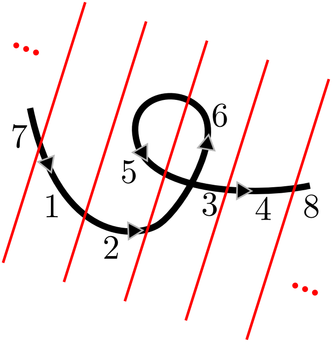

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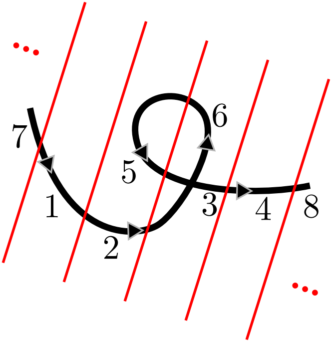

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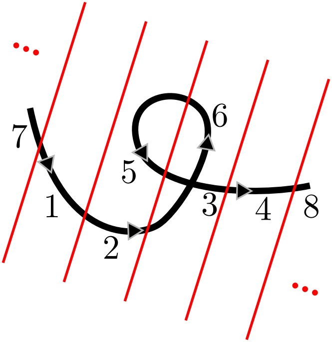

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$$L^{x,t,\mu}(\gamma) = \sum_{n=1}^{\infty} \frac{\|\gamma(C_n)\|_t}{2^n} \qquad \mathbf{len}(\gamma) = \int_0^1 \int_0^1 \int_0^1 L^{x,t,\mu}(\gamma)\, dx\, dt\, d\mu.$$

Compare with Cannon, Conner, Zastrow (2002) and Morse (1936).

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The function **len** can be defined analogously for any map γ from a locally connected continuum X to $\mathbb{R}^n$.

Parameterization by **len**

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Corollary

A set $\mathcal{F} \subseteq \Pi$ is closed (respectively, compact) if and only if $\widetilde{\mathcal{F}} = \{\widetilde{\gamma} : [\gamma] \in \mathcal{F}\}$ is a closed (respectively, compact) subset of $\mathcal{C}[0,1]$.

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Theorem

$\overline{\mathcal{F}}$ is compact if and only if:

1. *the set $\{\gamma(0) : [\gamma] \in \mathcal{F}\}$ is bounded; and*
2. *$\mathcal{F}$ satisfies the property (†).*

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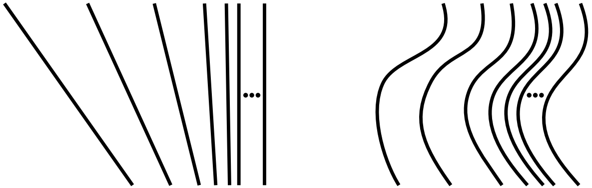

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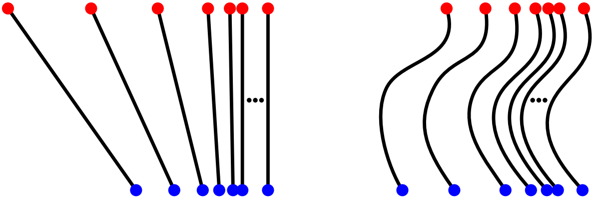

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