HIERARCHIES OF CHAOTIC MAPS ON CONTINUA

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Abstract. Let $f : X \longrightarrow X$ be a map of a continuum. In this paper we examine the following dynamical conditions on f:

(1) f is continuum-wise fully expansive
(2) f is weakly continuum-wise fully expansive
(3) f is mixing
(4) f is weakly mixing.

We first show that 1) implies 2), 2) implies 3) and 3) implies 4). Then we investigate what topological conditions will force the reverse implications to hold and give examples of when the reverse conditions do not hold. In particular, a map of the universal dendrite is given that is weakly mixing but not mixing.

1. Introduction and Definitions

By a *continuum*, we mean a compact connected metric space, and by a *map* we mean a continuous function.

One of the more interesting types of question in dynamics is *If $f : X \longrightarrow X$ is a map that has dynamic property A, under what conditions must f also have dynamic property B?* Some recent results can be found in [5] and [2]. The goal is to examine the equivalence and non-equivalence of the following dynamical properties on f: fully continuum-wise expansive, weakly fully continuum-wise expansive, mixing and weakly mixing under certain topological properties of continuum X. In doing so, we will construct a function on a dendrite that is weakly mixing, but not mixing.

Let $f : X \longrightarrow X$ be an onto map of a continuum X. The map f is *continuum-wise expansive* if there exists a $c > 0$ such that for every nondegenerate subcontinuum $Y \subset X$, there is an $n \in \mathbb{N}$ such that $\text{diam}(f^n(Y)) \geq c$. The map f is *continuum-wise fully expansive* if for every $\epsilon > 0$ and subcontinuum Y, there exists $N = N(\epsilon, Y)$ such that $\text{d}_H(f^n(Y), X) < \epsilon$ for all $n \geq N$, where d_H is the *Hausdorff distance.* Let

$$C(X) = \{Y|Y \text{ is a subcontinuum of } X\}$$

be the *hyperspace of subcontinua of* X. Let $c > 0$ and $E_c(f), E_F(f) \subset C(X)$ be defined by

$$E_c(f) = \{Y \in C(X)| \text{ there exists } n \in \mathbb{N} \text{ such that } \text{diam}(f^n(Y)) \geq c\},$$

$$E_F(f) = \{Y \in C(X)| \text{ for every } \epsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that } \text{d}_H(f^n(Y), X) < \epsilon \text{ for every } n \geq N\}.$$

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If $E_c(f)$ is dense in $C(X)$, then f is *weakly continuum-wise expansive.* If $E_F(f)$ is dense in $C(X)$, then f is *weakly continuum-wise fully expansive.* We define f to be *mixing* if for every pair of non-empty open sets U, V of X, there exists an M such that $f^m(U) \cap V \neq \emptyset$ for all $m \geq M$. A map $f : X \longrightarrow X$ is *transitive* if for any two non-empty open sets, $U, V \subset X$, there exists $n \in \mathbb{N}_0$ such that $f^n(U) \cap V \neq \emptyset$. This is equivalent to f having a *dense orbit*; that is $\{f^n(x)\}_{n=0}^{\infty}$ is dense in X for some $x \in X$.

Let $X^n = \Pi_{i=1}^n X$ and for $f : X \longrightarrow X$ let $\widehat{f}_n : X^n \longrightarrow X^n$ be the *induced map* of f on X^n defined by $\widehat{f}_n(x_1, ..., x_n) = (f(x_1), ..., f(x_n))$. Define $\pi_i : X^n \longrightarrow X$ to be the i-th projection (or coordinate) map. We say that f is *topologically n-transitive* if $\widehat{f}_n$ is transitive. A map f is *weakly mixing* if it is topologically 2-transitive. The following theorem by Furstenberg show the equivalence of these ideas on compact sets:

Theorem 1. [3], [4] *If $f : X \to X$ is a weakly mixing map of a compact space, then f is n-transitive for each n.*

Corollary 2. *If $f : X \to X$ is a weakly mixing map of a compact space, then for every non-empty open set V, there exists a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} d_H(f^{n_i}(V), X) = 0$*

Proof. Let $\mathcal{U}_i$ be a finite open cover of X with mesh less than $1/i$. Since f is $|\mathcal{U}_1|$-transitive, there exists n_1 such that $f^{n_1}(V) \cap U \neq \emptyset$ for each $U \in \mathcal{U}_1$. Hence $\mathrm{d}_H(f^{n_1}(V), X) < 1$. Continuing inductively, suppose that $n_1 < ... < n_i$ have been found. Since f is $|\mathcal{U}_{i+1}|$-transitive, there exists $n_{i+1} > n_i$ such that $f^{n_{i+1}}(V) \cap U \neq \emptyset$ for each $U \in \mathcal{U}_{i+1}$. Hence $\mathrm{d}_H(f^{n_{i+1}}(V), X) < 1/(i+1)$. □

On the other hand, Lau and Zame [6] gave an example of a map $\alpha : C \longrightarrow C$ of the Cantor set C that is weakly mixing, but not mixing.

Our goal is two-fold: First we will prove the following theorem establishing a hierarchy of chaotic maps:

Theorem 3. *If $f : X \longrightarrow X$ is a map of a continuum then:*

$$\begin{aligned} f \text{ is continuum-wise fully expansive} \quad &\Rightarrow \quad f \text{ is weakly continuum-wise fully expansive} \\ &\Rightarrow \quad f \text{ mixing} \\ &\Rightarrow \quad f \text{ weakly mixing.} \end{aligned}$$

Second and more importantly, we will examine what topological conditions will force the reverse implications to hold and give examples of when the reverse conditions do not hold.

2. Preliminary results

In this section we prove Theorem 3 and some other results that will be useful later. Throughout this section, we will assume that X and Y are arbitrary continua.

Proposition 4. *$f : X \longrightarrow X$ is mixing if and only if for every non-empty open set U, $\lim_{k\to\infty} d_H(f^k(U), X) = 0$.*

Proof. Suppose f is mixing, U is a non-empty open set and $\epsilon > 0$. Since X is compact, there exists a finite subcover $\{B_{\epsilon/2}(x_i)\}_{i=1}^n$ of $\epsilon/2$ balls. Then, there exists $K > 0$ such that $f^k(U) \cap B_{\epsilon/2}(x_i) \neq \emptyset$ for all $k \geq K$ and i. Therefore $\mathrm{d}_H(f^k(U), X) < \epsilon$ for all $k \geq K$.

Conversely, suppose that U, V are non-empty open sets in X. Then there exists $\epsilon > 0$ and $v \in V$ such that $B_\epsilon(v) \subset V$. Let $K > 0$ be such that $\mathrm{d}_H(f^k(U), X) < \epsilon$ for all $k \geq K$. Then it follows that

$$f^k(U) \cap V \supseteq f^k(U) \cap B_\epsilon(v) \neq \emptyset$$

for all $k \geq K$. □

Proposition 5. *If $f : X \longrightarrow X$ is weakly continuum-wise fully expansive, then f is mixing.*

Proof. Let U, V be non-empty open sets of X. Then there is $Z \in E_F(f)$ such that $Z \subset U$. Let $v \in V$ and $\epsilon > 0$ such that $B_\epsilon(v) \subset V$. Since f is weakly continuum-wise fully expansive, there exists an N such that $\mathrm{d}_H(f^j(Z), X) < \epsilon/2$ for all $j \geq N$. Then $f^j(U) \cap V \neq \emptyset$ for all $j \geq N$. □

Proposition 6. *Suppose that $f : X \longrightarrow X$ and that for any non-empty open set $U \subset X$ there exists an increasing sequence $\{n_i\}_{i=1}^\infty$ such that $d_H(f^{n_i}(U), X) \to 0$ as $i \to \infty$. Then f is weakly mixing.*

Proof. Let U_1, U_2, V_1, V_2 be non-empty open sets in X. Choose $n_i \geq 0$ such that $f^{n_i}(U_1) \cap U_2 \neq \emptyset$. Let $W = U_1 \cap f^{-n_i}(U_2)$. Then W is a non-empty open set, so we can choose n_j such that $f^{n_j}(W) \cap V_1 \neq \emptyset$ and $f^{n_j}(W) \cap f^{-n_i}(V_2) \neq \emptyset$. It follows that $f^{n_j}(U_1) \cap V_1 \neq \emptyset$ and $f^{n_j}(U_2) \cap V_2 \neq \emptyset$. □

Since mixing clearly implies weakly mixing and fully continuum-wise expansive clearly implies weakly fully continuum-wise expansive, Theorem 3 follows. From here on out the paper will focus on the more interesting results on the reverse implications.

The following lemmas and propositions will be useful later:

Lemma 7. *Suppose $f : X \longrightarrow X$, S is a subset of X and $\{n_i\}_{i=1}^\infty$ is an increasing sequence such that*

1. $\lim_{i\to\infty} d_H(f^{n_i}(S), X) = 0$
2. *there exists a n such that $S \subset f^n(S)$.*

Then $\lim_{n\to\infty} d_H(f^n(S), X) = 0$.

Proof. Since X is compact, f is uniformly continuous. Let $\epsilon > 0$ and $U \subset X$. Then there exists $\delta > 0$ such that if $\mathrm{d}_H(U, X) = \mathrm{d}_H(\overline{U}, X) < \delta$ then

$$\mathrm{d}_H(f^j(U), X) = \mathrm{d}_H(f^j(\overline{U}), X) = \mathrm{d}_H(f^j(\overline{U}), f^j(X)) < \epsilon$$

for every $j \in \{0, ..., n\}$. Since $\lim_{i\to\infty} \mathrm{d}_H(f^{n_i}(S), X) = 0$, there exists α such that $\mathrm{d}_H(f^{n_\alpha}(S), X) < \delta$. Since $S \subset f^n(S)$, $f^{n_\alpha}(S) \subset f^{n_\alpha + kn}(S)$ for all $k \in \mathbb{N}_0$. Thus $\mathrm{d}_H(f^{n_\alpha+kn}(S), X) < \delta$ for all $k \in \mathbb{N}_0$. Hence, $\mathrm{d}_H(f^{n_\alpha+kn+j}(S), X) < \epsilon$ for all $k \in \mathbb{N}_0$ and $j \in \{0, ..., n\}$. If $m \geq n_\alpha$ then $m = n_\alpha + kn + j$ for some $k \in \mathbb{N}_0$ and $j \in \{0, ..., n\}$. Thus $\mathrm{d}_H(f^m(S), X) < \epsilon$ for all $m \geq n_\alpha$. □

Lemma 8. *Suppose that $f : X \longrightarrow X$ is weakly mixing and there exist a collection of open sets $\{U_\alpha\}_{\alpha\in\Omega}$ such that*

1. *For every $\alpha \in \Omega$, there exists n_α such that $U_\alpha \subset f^{n_\alpha}(U_\alpha)$*
2. *For every non-empty open set V, there exists $\alpha \in \Omega$ and m_α such that $U_\alpha \subset f^{m_\alpha}(V)$.*

Then f is mixing.

Proof. Let V be a non-empty open set and let α be such that $U_\alpha \subset f^{m_\alpha}(V)$ for some m_α. Then $f^n(U_\alpha) \subset f^{m_\alpha+n}(V)$ for all $n \in \mathbb{N}_0$. By Lemma 7, $\lim_{n\to\infty} \mathrm{d}_H(f^n(U_\alpha), X) = 0$. Hence it follows that $\lim_{n\to\infty} \mathrm{d}_H(f^{m_\alpha+n}(V), X) = 0$ and f is mixing by Proposition 4. □

Proposition 9. *Suppose that $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$ are both mixing. Then $F : X \times Y \longrightarrow X \times Y$ defined by $F(x, y) = (f(x), g(x))$ is mixing.*

Proof. Let U, V be non-empty open subsets of $X \times Y$. Let U_X, V_X be non-empty open sets of X and U_Y, V_Y be non-empty open sets of Y such that $U_X \times U_Y \subset U$ and $V_X \times V_Y \subset V$. Since f and g are mixing there exists $N > 0$ such that $f^n(U_X) \cap V_X \neq \emptyset$ and $g^n(U_Y) \cap V_Y \neq \emptyset$ for all $n \geq N$. Hence $F^n(U) \cap V \supseteq (f^n(U_X) \cap V_X) \times (g^n(U_Y) \cap V_Y) \neq \emptyset$ for all $n \geq N$. □

Proposition 10. *Suppose that $f : X \longrightarrow X$ is mixing and $g : Y \longrightarrow Y$ is weakly mixing. Then $F : X \times Y \longrightarrow X \times Y$ defined by $F(x, y) = (f(x), g(x))$ is weakly mixing.*

Proof. Let U, V, U_X, V_X, U_Y, V_Y be defined as in Proposition 9. There exists an $N > 0$ such that $f^n(U_X) \cap V_X \neq \emptyset$ for all $n \geq N$. Since g is weakly mixing, for every $M > N$ there exists $m \geq M$ such that $g^m(U_Y) \cap V_Y \neq \emptyset$. Hence, it follows that $F^m(U) \cap V \supseteq (f^m(U_X) \cap V_X) \times (g^m(U_Y) \cap V_Y) \neq \emptyset$. □

Proposition 11. *Suppose that $f : X \longrightarrow X$ is any map and $g : Y \longrightarrow Y$ is not mixing. Then $F : X \times Y \longrightarrow X \times Y$ defined by $F(x, y) = (f(x), g(x))$ is not mixing.*

Proof. Since g is not mixing there exist non-empty open sets U, V of Y and a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ such that $g^{n_i}(U) \cap V = \emptyset$. Then $F^{n_i}(X \times U) \cap (X \times V) = \emptyset$ for all i. □

3. Results on locally connected continua

In this section we show that in general, the only implication of Theorem 3 that can be reversed for locally connected continua is mixing implying weakly continuum-wise fully expansive. However for graph continua, all of the implications in Theorem 3 can be reversed.

Proposition 12. *If $f : X \longrightarrow X$ is mixing and X is locally connected, then f is weakly continuum-wise fully expansive.*

Proof. Let $\mathcal{H}$ be the collection of subcontinua of X that have nonempty interior. Then $\mathcal{H}$ is dense in $C(X)$. Since f is mixing,

$$\lim_{n\longrightarrow\infty} \mathrm{d}_H(f^n(Y), X) = \lim_{n\longrightarrow\infty} \mathrm{d}_H(f^n(Y^\circ), X) = 0$$

for all $Y \in \mathcal{H}$. Hence, f is weakly continuum-wise fully expansive. □

The following corollary may be of interest to those who study complex dynamics:

Corollary 13. *Suppose that $r : \mathbb{C} \longrightarrow \mathbb{C}$ is a rational map with locally connected Julia set J. Then $f|_J$ is weakly continuum-wise fully expansive.*

Arc $[a, b] \subset X$ is a *free-arc* if $[a, b]^\circ = (a, b)$. The following theorem has also been shown by Haranczyk, Kwietniak, and Oprocha in [5] and more general results appear in [2]. For completeness and reference in later proofs, our version of the proof is given here.

Lemma 14. *Suppose that $f : X \longrightarrow X$ is weakly mixing and X is locally connected with a free arc. Then f is mixing.*

Proof. Let $[a, b]$ be a free arc in X. Choose $a < x_1 < y_1 < x_2 < y_2 < b$. Then there exists n, m such that

$$(a, x_1) \cap f^n((x_1, y_1)) \neq \emptyset,$$
$$(y_1, x_2) \cap f^n((x_1, y_1)) \neq \emptyset,$$
$$(a, x_1) \cap f^m((x_2, y_2)) \neq \emptyset,$$
$$(y_1, x_2) \cap f^m((x_2, y_2)) \neq \emptyset.$$

Therefore one of the following must occur:

(1) $(x_1, y_1) \subset f^n((x_1, y_1))$ and $(x_2, y_2) \subset f^m((x_2, y_2))$
(2) $(x_1, y_1) \subset f^n((x_1, y_1))$ and $(x_1, y_1) \subset f^m((x_2, y_2))$
(3) $(x_2, y_2) \subset f^n((x_1, y_1))$ and $(x_2, y_2) \subset f^m((x_2, y_2))$
(4) $(x_2, y_2) \subset f^n((x_1, y_1))$ and $(x_1, y_1) \subset f^m((x_2, y_2))$.

Notice that 4) implies that $(x_1, y_1) \subset f^{n+m}((x_1, y_1))$ and $(x_2, y_2) \subset f^{m+n}((x_2, y_2))$. Let U be any non-empty open set in X. Then there exists a connected open set $V \subset U$ and a natural number k such that

$$(a, x_1) \cap f^k(V) \neq \emptyset \text{ and } (y_1, x_2) \cap f^k(V) \neq \emptyset.$$

Thus one of the following must be true:

(1) $(x_1, y_1) \subset f^k(U)$
(2) $(x_2, y_2) \subset f^k(U)$.

Hence, it follows from Lemma 8 that f is mixing. □

Theorem 15. *Suppose that $f : G \longrightarrow G$ is a map of a graph continuum G. Then the following are equivalent:*

(1) *f is weakly mixing*
(2) *f is mixing*
(3) *f is weakly continuum-wise fully expansive*
(4) *f is continuum-wise fully expansive.*

Proof. 4) ⇒ 3) ⇒ 2) ⇒ 1) is just Theorem 3. Also, 1) ⇒ 3) follows from Proposition 12 and Lemma 14. To show that 3) ⇒ 4), notice that since every subcontinuum of a graph has non-empty interior, f must be continuum-wise fully expansive. □

Let

$$C_\epsilon(f) = \{Y \in C(X) | \text{ there exists } N \in \mathbb{N} \text{ such that } \mathrm{d}_H(f^n(Y), X) \geq \epsilon \text{ for every } n \geq N\}.$$

The following is a simple example of a map on a dendrite that is weakly continuum-wise fully expansive but not continuum-wise fully expansive (or even continuum-wise expansive). A *dendrite* is a locally connected continuum that contains no simple closed curve. If D is a dendrite and $p \in D$, then the *order of p*, denoted by $\mathrm{ord}(p, D)$ is the number of components of $D - \{p\}$. Points of order greater than or equal to 3 are called *ramification points* of D. The *standard universal dendrite D_m of order $m \in \mathbb{N} \cup \{\omega\}$* is a dendrite with the following properties:

(1) every ramification point of D_m has order m
(2) for every arc A contained in D_m, the set of all ramification points that are contained in A is a dense subset of A.

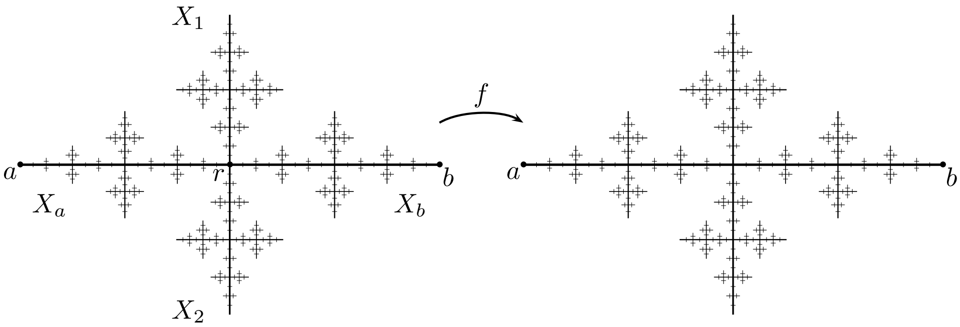


FIGURE 1. The map $f : D_4 \to D_4$. Each of the subdendrites X_a, X_b, X_1, and X_2 is mapped homeomorphically onto all of D_4, with $f(r) = a$.

Example 16. *There exists a map $f : D_4 \longrightarrow D_4$ and $0 < \epsilon$ such that $E_F(f)$ and $C_\epsilon(f)$ are dense in $C(X)$ where D_4 is the universal dendrite of order 4.*

Proof. Sketch of proof. Let $a, b \in D_4$ be endpoints and r be a ramification point of order 4 in $[a, b]$. Let X_a, X_b, X_1 and X_2 be the closure of the components of $D_4 - \{r\}$ such that $a \in X_a$ and $b \in X_b$. By Proposition 3.5 in [1], there exist homeomorphisms $h_1 : X_1 \longrightarrow D_4$, $h_2 : X_2 \longrightarrow D_4$, $h_a : X_a \longrightarrow D_4$, and $h_b : X_b \longrightarrow D_4$ such that $h_1(r) = h_2(r) = h_a(r) = h_b(r) = a$ and $h_a(a) = h_b(b) = b$. Define

$$f(x) = \begin{cases} h_1(x) & \text{if } x \in X_1 \\ h_2(x) & \text{if } x \in X_2 \\ h_a(x) & \text{if } x \in X_a \\ h_b(x) & \text{if } x \in X_b. \end{cases}$$

See Figure 1. Then it can be checked that every non-empty open set eventually maps to all of D_4. It follows that f weakly continuum-wise fully expansive by Proposition 12.

Let $\epsilon > 0$ be such that $\mathrm{d}_H([a, b], D_4) \geq \epsilon$. If Y is a subcontinuum of D_4 that contains no endpoint of D_4, then there exists an N such that $f^n(Y) \subseteq [a, b]$ for all $n \geq N$. Hence $Y \in C_\epsilon(f)$. It follows that $C_\epsilon(f)$ is dense in $C(X)$. □

A map $f : X \longrightarrow X$ has *sensitive dependence on initial conditions* if there exists $c > 0$ such that for every $x \in X$ and open set U that contains x, there exists $n \in \mathbb{N} \cup \{0\}$ and $y \in U$ such that $\mathrm{d}(f^n(x), f^n(y)) \geq c$.

Note: The induced map $\widetilde{f} : C(D_4) \longrightarrow C(D_4)$ on the hyperspace of D_4 has sensitive dependence on initial conditions.

Next we give an example of a map of the standard universal dendrite D_ω that is weakly mixing, but not mixing.

In the following, ω denotes the set of natural numbers (including 0), $\omega = \{0, 1, 2, \ldots\}$, and Q denotes the set of dyadic rational numbers in the interval $(0, 1)$, i.e. rational numbers of the form $\frac{p}{2^q}$ where p is odd, $q \geq 1$, and $0 < p < 2^q$. Given $k \in \omega$, let $S_k = \omega \times (Q \times \omega)^k$, and let $S = \{\emptyset\} \cup \bigcup_{k \in \omega} S_k$.

Given a finite sequence s and an integer $n \geq 0$ smaller than the length of s, denote by $s\upharpoonright n$ the sequence consisting of the first n elements of s. We write $s \sqsubset s'$ to mean there is an integer n such that $s = s'\upharpoonright n$. Denote the concatenation of the sequences s and s' by $s\frown s'$.

We view the universal dendrite D_ω as being constructed from a recursively defined "skeleton" of arcs in the plane as follows. First, to a single point $o \in \mathbb{R}^2$ we attach countably many arcs $A[\langle m\rangle]$, $m \in \omega$, of lengths $< 2^{-m}$. Then, having defined $A[s]$ for all $s \in S_k$ (for some $k \in \omega$), for each dyadic rational $\frac{p}{2^q} \in Q$ we attach countably many arcs $A[s\frown\langle \frac{p}{2^q}, m\rangle]$, $m \in \omega$, to the point at position $\frac{p}{2^q}$ along $A[s]$, of lengths $< 2^{(-q-m-m_0-\sum(q_i+m_i))}$. Then $D_\omega = \overline{\bigcup_{s\in S} A[s]}$ (here the closure is taken in $\mathbb{R}^2$). See Figure 2 below for an illustration.

Given $j \in \omega$, let E_j be the subdendrite $E_j = \overline{\bigcup_{s \sqsupseteq \langle j\rangle} A[s]}$.

Observe that one endpoint of each arc $A[s]$ is a branch point of D_ω and the other is an endpoint of D_ω. Given $s_1, s_2 \in S \setminus \{\emptyset\}$, the notation $A[s_1] \overset{f}{\mapsto} A[s_2]$ means that the function f maps $A[s_1]$ linearly onto $A[s_2]$, sending the endpoint of $A[s_1]$ which is an endpoint of D_ω to the endpoint of $A[s_2]$ which is an endpoint of D_ω. By $A[s] \overset{f}{\mapsto} \{o\}$, we mean f is constant on $A[s]$, and $f(A[s]) = \{o\}$.

Define $A[\emptyset] = \{o\}$.

Example 17. *There is a continuous function $f : D_\omega \to D_\omega$ which is weakly mixing but not mixing. In fact, given any non-empty open subset $U \subset D_\omega$ which is contained in a single subdendrite E_j, U will have forward images which have arbitrarily small diameters, and forward images which are arbitrarily close to the whole space D_ω (in the Hausdorff distance).*

Proof. Given an infinite set $Z \subseteq \omega$, we will construct a continuous function $f_Z : D_\omega \to D_\omega$.

We will define functions $\alpha = \alpha_Z : S \setminus \{\emptyset\} \to S$ and $n = n_Z : S \setminus \{\emptyset\} \to \omega$ simultaneously by recursion on the length of $s \in S$. The function α will dictate the images of the arcs $A[s]$ under the map f_Z. The construction of α will be such that each $s \in S$ will eventually map forward under some iterate of α to $\emptyset$; $n(s)$ will be the smallest integer $n \geq 0$ such that $\alpha^{n+1}(s) = \emptyset$.

(1) For $s \in S_0$:
- $\alpha(\langle 0\rangle) = \emptyset$ and $n(\langle 0\rangle) = 0$;
- If $m \geq 1$, $\alpha(\langle m\rangle) = \langle m-1\rangle$ and $n(\langle m\rangle) = n(\alpha(\langle m\rangle)) + 1 = m$.

(2) For $s \in S_1$:
- If $s = \langle 0, \frac{p_1}{2^{q_1}}, m_1\rangle$, define $\alpha(s) = \langle m\rangle$, where m is the (q_1+m_1)-th element of Z;
- If $s = \langle m_0, \frac{p_1}{2^{q_1}}, m_1\rangle$ where $m_0 \geq 1$, define $\alpha(s) = \langle m_0 - 1, \frac{p_1}{2^{q_1}}, m_1\rangle$.

In either case, define $n(s) = n(\alpha(s)) + 1$.

(3) For $s \in S_k$ $(k \geq 2)$:
- If $s = \langle 0, \frac{p_1}{2^{q_1}}, m_1, \ldots, \frac{p_k}{2^{q_k}}, m_k\rangle$, define

$$\alpha(s) = \alpha\left(\left\langle 0, \frac{p_1}{2^{q_1}}, m_1, \ldots, \frac{p_{k-1}}{2^{q_{k-1}}}, m_{k-1}\right\rangle\right) \frown \left\langle \frac{p_k}{2^{q_k}}, m\right\rangle$$

where m is the (q_k+m_k)-th integer ≥ 0 such that

$$n\left(\ \alpha\left(\left\langle 0, \frac{p_1}{2^{q_1}}, m_1, \ldots, \frac{p_{k-1}}{2^{q_{k-1}}}, m_{k-1}\right\rangle\right) \frown \left\langle \frac{p_k}{2^{q_k}}, m\right\rangle\ \right) \in Z;$$

- If $s = \langle m_0, \frac{p_1}{2^{q_1}}, m_1, \dots, \frac{p_k}{2^{q_k}}, m_k \rangle$ where $m_0 \geq 1$, define
$$\alpha(s) = \left\langle m_0 - 1, \frac{p_1}{2^{q_1}}, m_1, \dots, \frac{p_k}{2^{q_k}}, m_k \right\rangle.$$
Again in either case, define $n(s) = n(\alpha(s)) + 1$.

Note that in part (3), the sequence $\alpha\left(\left\langle 0, \frac{p_1}{2^{q_1}}, m_1, \dots, \frac{p_{k-1}}{2^{q_{k-1}}}, m_{k-1} \right\rangle\right)$ has length $2k-3$, so that $n\left(\alpha\left(\left\langle 0, \frac{p_1}{2^{q_1}}, m_1, \dots, \frac{p_{k-1}}{2^{q_{k-1}}}, m_{k-1} \right\rangle\right) \frown \left\langle \frac{p_k}{2^{q_k}}, m \right\rangle \right)$ is defined for all m by recursion.

If no such integer m exists in part (3) of the above recursion for some s, then we say that $\alpha = \alpha_Z$ is undefined at s. Observe that if $\alpha(s)$ is defined, then in unwinding the above recursion, the value $\alpha(s)$ depends only on some finite initial segment of the set Z. Therefore, we have:

Claim 17.1. *If $s \in S$ is such that $\alpha_Z(s)$ is defined, then there exists some integer $N > 0$ such that if $Z' \subseteq \omega$ is any other set with $Z' \cap [0, N] = Z \cap [0, N]$, then $\alpha_{Z'}(s) = \alpha_Z(s)$.*

An easy induction shows that

$$(*) \qquad n\left(\left\langle m_0, \frac{p_1}{2^{q_1}}, m_1, \dots, \frac{p_k}{2^{q_k}}, m_k \right\rangle\right) \geq m_0 + \sum_{i=1}^{k} (q_i + m_i).$$

Claim 17.2. *For each $s \in S$ and $N \in \omega$, there exists $M \geq 0$ such that if $[N, N + M] \subset Z$, then $\alpha(s)$ is defined.*

Proof of Claim 17.2. We proceed by induction on the length of s.

Clearly $\alpha\left(\left\langle 0, \frac{p_1}{2^{q_1}}, m_1 \right\rangle\right)$ is defined as long as Z has at least $q_1 + m_1$ elements. Furthermore, for any $k \geq 1$ and $m_0 \in \omega$, $\alpha\left(\left\langle m_0, \frac{p_1}{2^{q_1}}, m_1, \dots, \frac{p_k}{2^{q_k}}, m_k \right\rangle\right)$ is defined as soon as $\alpha\left(\left\langle 0, \frac{p_1}{2^{q_1}}, m_1, \dots, \frac{p_k}{2^{q_k}}, m_k \right\rangle\right)$ is defined.

Consider $s = \left\langle 0, \frac{p_1}{2^{q_1}}, m_1, \dots, \frac{p_k}{2^{q_k}}, m_k \right\rangle$, where $k \geq 2$. By induction, we can find $M_0 > 0$ large enough so that

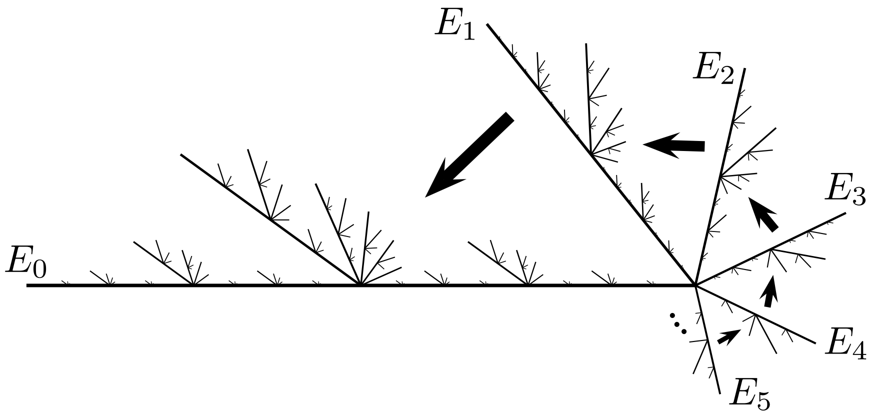
(1) if $[N, N + M_0] \subset Z$, then $\alpha\left(\left\langle 0, \frac{p_1}{2^{q_1}}, m_1, \dots, \frac{p_{k-1}}{2^{q_{k-1}}}, m_{k-1} \right\rangle\right)$ is defined.

Assume $[N, N + M_0] \subset Z$. For $m \in \omega$, let
$$s_m = \alpha\left(\left\langle 0, \frac{p_1}{2^{q_1}}, m_1, \dots, \frac{p_{k-1}}{2^{q_{k-1}}}, m_{k-1} \right\rangle\right) \frown \left\langle \frac{p_k}{2^{q_k}}, m \right\rangle.$$
Again by induction, we can find $M_1 \geq M_0$ large enough so that

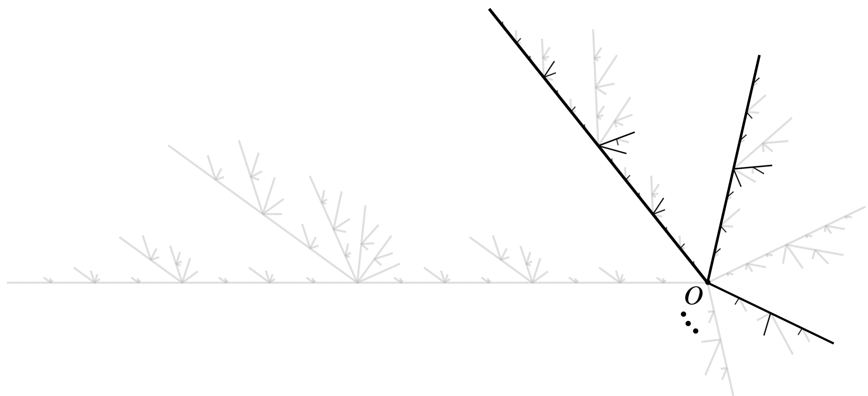
(2) if $[N, N + M_1] \subset Z$, then $\alpha(s_m)$ is defined for all $N \leq m \leq N + q_k + m_k$.

Assume $[N, N + M_1] \subset Z$. For each number m with $N \leq m \leq N + q_k + m_k$, we have $n(s_m) \geq N$ by $(*)$. Let $M_2 \geq M_1$ be larger than all the numbers $n(s_m)$ for $N \leq m \leq N + q_k + m_k$. Then if $[N, N + M_2] \subset Z$, there are at least $q_k + m_k$ values of m for which $n(s_m) \in Z$. It follows that $\alpha(s)$ is defined if $[N, N + M_2] \subset Z$. □(Claim 17.2)

Observe that if $s \sqsubset s'$, then $\alpha(s) \sqsubset \alpha(s')$. Moreover, $|\alpha(s')| - |\alpha(s)| = |s'| - |s|$ whenever $|s| \geq 3$. It follows that the relation $A[s] \overset{f_Z}{\mapsto} A[\alpha(s)]$ defines a continuous function $f_Z : D_\omega \to D_\omega$. See Figure 2 for an illustration of the action of the map f_Z on D_ω.



(a) For $j > 0$, the subdendrite E_j is mapped one-to-one onto E_{j-1}.



(b) A rough depiction of the image of the subdendrite E_0. The arc $A[\langle 0\rangle]$ is mapped to the point o, and the copies of D_ω attached to $A[\langle 0\rangle]$ are mapped into a carefully selected subdendrite (indicated in black), defined using the set Z.

FIGURE 2. The map $f_Z : D_\omega \to D_\omega$.

Recall that given $j \in \omega$, E_j denotes the subdendrite $E_j = \overline{\bigcup_{s \sqsupseteq \langle j\rangle} A[s]}$. The recursive definition of the function α is specifically designed so that the following claim holds:

Claim 17.3. *For all $n > 0$,*

$$f_Z^n(E_0) \subseteq \bigcup_{n+j-1 \in Z} E_j.$$

Proof. Given $s = \langle 0, \frac{p_1}{2^{q_1}}, m_1, \ldots, \frac{p_k}{2^{q_k}}, m_k\rangle$, recall that $n(s)$ is the smallest $n \geq 0$ such that $\alpha^{n+1}(s) = \emptyset$, i.e. the smallest $n \geq 0$ such that $f^{n+1}(A[s]) = \{o\}$. Suppose for induction that the claim holds for all s' with $n(s') < n(s)$.

For all $n > n(s)$, we have $f^n(A[s]) = \{o\}$. If $n < n(s)$ and $\alpha^n(s)$ has length > 1, then there is some $s' \sqsubset s$ with $\alpha^n(s') \sqsubset \alpha^n(s)$, $\alpha^n(s') \neq \emptyset$. It follows by induction that $f^n(A[s']) \subset E_j$ where $n + j - 1 \in Z$, hence $f^n(A[s]) \subset E_j$ as well.

If $\alpha^n(s) = \langle j \rangle$, then $f^n(A[s]) \subset E_j$ and $j + n = n(s)$. Recall that $n(s) - 1 = n(\alpha(s)) \in Z$ by definition of n and α. Therefore $j + n - 1 \in Z$, as desired. □

Claim 17.4. *Suppose that for any $M > 0$ there exists $N \in \omega$ such that $[N, N + M] \cap Z = \emptyset$. Then for any $\varepsilon > 0$, there exists $n > 0$ such that $f_Z^n(E_0)$ has diameter $< \varepsilon$.*

Proof of Claim 17.4. Let $M > 0$ be large enough so that $\bigcup_{m \geq M} E_m$ has diameter $< \varepsilon$. Let $N \in \omega$ be such that $[N, N + M] \cap Z = \emptyset$. Then by Claim 17.3, we have $f_Z^{N+1}(E_0) \subseteq \bigcup_{m \geq M} E_m$, and so $f_Z^{N+1}(E_0)$ has diameter $< \varepsilon$. □(Claim 17.4)

We next aim to show that if Z also contains very long strings of consecutive integers, then any non-empty open set will grow arbitrarily large under iteration of f_Z. To this end, we introduce some further notation.

For $k \in \omega$, let $S_k^- = (\omega \times Q)^k$, and let $S^- = \bigcup_{k \in \omega} S_k^-$. Given $s \in S^-$ and $K, m \geq 0$, let $T_{s,K,m}$ be the set

$$T_{s,K,m} = \Bigg\{ s \frown \left\langle m_0', \frac{p_1'}{2^{q_1'}}, m_1', \ldots, \frac{p_k'}{2^{q_k'}}, m_k' \right\rangle : \ 0 \leq k \leq K, \\ m \leq m_0' \leq K + m, \text{ and for each } 0 < i \leq k \text{ we have} \\ 1 \leq q_i' \leq K, \ 1 \leq p_i' < 2^{q_i'}, \text{ and } 0 \leq m_i' \leq K \Bigg\}$$

The set $A[T_{s,K,m}] = \bigcup \{A[s] : s \in T_{s,K,m}\}$ is a tree in D_ω.

Note in particular that $A[T_{\emptyset,K,0}] \to D_\omega$ as $K \to \infty$.

Claim 17.5. *For any $N, K \geq 0$ and any $s \in S^-$, there exist $M, m, n \geq 0$ such that if $[N, N + M] \subset Z$, then $\alpha^n(T_{s,K,m}) = T_{\emptyset,K,0}$.*

Proof of Claim 17.5. We prove this claim using the following subclaim.

Given $s = \langle m_0, \frac{p_1}{2^{q_1}}, \ldots, m_k, \frac{p_{k+1}}{2^{q_{k+1}}} \rangle \in S^-$, define $\alpha(s) = \alpha(\langle m_0, \frac{p_1}{2^{q_1}}, \ldots, m_k \rangle) \frown \langle \frac{p_{k+1}}{2^{q_{k+1}}} \rangle$.

Subclaim 17.5.1. *For all $s \in S^-$ ($|s| \geq 2$), $K, N \geq 0$, and $m \geq N$, there exists $M \geq 0$ such that if $[N, N + M] \subset Z$ then $\alpha(T_{s,K,m}) = T_{\alpha(s),K,\tilde{m}}$, where $\tilde{m}$ is such that $\alpha(s \frown \langle m \rangle) = \alpha(s) \frown \langle \tilde{m} \rangle$.*

Proof of Subclaim 17.5.1. If $s \not\sqsupset \langle 0 \rangle$, the claim trivially holds. Therefore, assume $s \sqsupset \langle 0 \rangle$.

Let M be large enough so that $\alpha(s')$ is defined for all $s' \in T_{s,K,m}$. Since $m \geq N$, it follows from the observation $(*)$ that $n(s') \geq N$ for all $s' \in T_{s,K,m}$. Then provided M is chosen large enough, we have

$$n\left(\alpha\left(s \frown \left\langle m_0', \frac{p_1'}{2^{q_1'}}, m_1', \ldots, \frac{p_{i-1}'}{2^{q_{i-1}'}}, m_{i-1}' \right\rangle \right) \frown \left\langle \frac{p_i'}{2^{q_i'}}, m \right\rangle \right) \in [N, N + M] \subset Z$$

for each $1 \leq i \leq K$ and all $m \leq K$. Therefore, by definition of the function α,

$$\alpha\left(s \frown \left\langle m_0', \frac{p_1'}{2^{q_1'}}, m_1', \ldots, \frac{p_i'}{2^{q_i'}}, m_i' \right\rangle \right) = \alpha\left(s \frown \left\langle m_0', \frac{p_1'}{2^{q_1'}}, m_1', \ldots, \frac{p_{i-1}'}{2^{q_{i-1}'}}, m_{i-1}' \right\rangle \right) \frown \left\langle \frac{p_i'}{2^{q_i'}}, m_i' \right\rangle$$

for each $1 \leq i \leq K$ and all $m_i' \leq K$. The subclaim now follows by a simple induction on i. □(Subclaim 17.5.1)

Now to establish Claim 17.5, fix $N, K \geq 0$, let $s = \langle m_0, \frac{p_1}{2^{q_1}}, \ldots, m_k, \frac{p_{k+1}}{2^{q_{k+1}}}\rangle \in S^-$, and let $s' = \langle m_0, \frac{p_1}{2^{q_1}}, \ldots, m_k\rangle \in S$. By applying Subclaim 17.5.1 repeatedly, we can choose $m \geq N$ and M both large enough so that if $[N, N+M] \subset Z$, then $\alpha^i(T_{s,K,m}) = T_{\alpha^i(s),K,\tilde{m}_i}$ for all $n < n(s')$, where each $\tilde{m}_i$ is $\geq N$. In particular, $\tilde{m}_{n(s')-1} \geq N$, so $\alpha^{n(s')}(T_{s,K,m}) = \alpha(T_{\alpha^{n(s')-1}(s),K,\tilde{m}_{n(s')-1}}) = T_{\emptyset,K,\tilde{m}}$ for some $\tilde{m}$. Let $n = n(s') + \tilde{m}$. Then $\alpha^n(T_{s,K,m}) = T_{\emptyset,K,0}$, as desired. □(Claim 17.5)

We are now ready to construct the set Z, which will be the union of a family of intervals in ω. Let $\{s_0, s_1, s_2, \ldots\}$ enumerate the set S^-.

Let $N_0 = M_0 = 0$. Having defined N_K and M_K, let $N_{K+1} = M_K + K$. Then let $M_K \geq N_K$ be large enough (Claims 17.2 and 17.5) so that if $[N_K, M_K] \subset Z$, then for each $i \leq K$ there is some $m, n \in \omega$ such that $\alpha(s)$ is defined for each $s \in T_{s_i,K,m}$, and $\alpha^n(T_{s_i,K,m}) = T_{\emptyset,K,0}$.

We now define $Z = \bigcup_{K=0}^{\infty}[N_K, M_K]$. Consider the map $f_Z : D_\omega \to D_\omega$. By Claim 17.4, the set E_0 will have forward images which are arbitrarily small. Since each subdendrite E_j is mapped onto E_0 in j iterates, it follows that any non-empty open set U which is contained in one of the sets E_j will have forward images which are arbitrarily small. Thus, in particular, f_Z is not mixing.

On the other hand, observe that for any non-empty open set $U \subseteq D_\omega$, there is some $s = s_i \in S^-$ such that $A[s'] \subset U$ for all $s' \in S$ with $s' \sqsupseteq s$. By construction, for all $K \geq i$, there is an m and an iterate of α_Z which maps $T_{s_i,K,m}$ to $T_{\emptyset,K,0}$. It follows that U has forward images under f_Z which are arbitrarily close to D_ω in the Hausdorff distance. So f_Z is weakly mixing by Proposition 6. □

4. Results on the Cantor fan

In this section, we show that none of the implications in Theorem 3 can be reversed for the Cantor fan.

Let $C = \Pi_{i=-\infty}^{\infty}\{0,1\}$ be the Cantor set, $X = ([0,1] \times C)/(\{0\} \times C)$ be the Cantor fan and for $c \in C$ define $[0,1]_c = [0,1] \times \{c\}$. Let $\pi_0 : X \longrightarrow [0,1]$ be the projection map defined by if $(x,c) \in [0,1] \times C$ then $\pi_0(x,c) = x$. Let $\sigma : C \longrightarrow C$ be the *shift map* on the Cantor set defined by $\sigma(\langle x_i\rangle_{i=-\infty}^{\infty}) = \langle x_{i+1}\rangle_{i=-\infty}^{\infty}$.

Example 18. *There exists a map on the Cantor fan that is weakly continuum-wise fully expansive, but not continuum-wise expansive.*

Let $\sigma : C \longrightarrow C$ be the shift homeomorphism. Then it is well known that σ is transitive. Define $f : X \longrightarrow X$ by

$$f(x,c) = \begin{cases} (x, \sigma(c)) & \text{if } \pi_0(c) = 0 \\ (3x, c) & \text{if } \pi_0(c) = 1 \text{ and } x \in [0,1/3] \\ (2-3x, c) & \text{if } \pi_0(c) = 1 \text{ and } x \in [1/3, 2/3] \\ (3x-2, \sigma(c)) & \text{if } \pi_0(c) = 1 \text{ and } x \in [0,1/3]. \end{cases}$$

Then $f\restriction_{[0,1]_\mathbf{0}} = id\restriction_{[0,1]_\mathbf{0}}$, where $\mathbf{0} = \langle 0\rangle_{i=-\infty}^{\infty}$ is a fixed point under σ. Thus, f is not continuum-wise expansive. However, let $t \in C$ such that $\{\sigma^n(t)\}_{n=0}^{\infty}$ is dense in X. Then it can be shown that

$$\lim_{n\to\infty} \mathrm{d}_H(f^n([a,b]_{\sigma^k(t)}), X) = 0$$

for and $0 \leq a < b \leq 1$ and $k \geq 0$. Since the set of continua $K \subseteq X$ which contain an arc of the form $[a,b]_{\sigma^k(t)}$ is dense in $C(X)$, it follows that f is weakly continuum-wise fully expansive.

Example 19. *There exists a map on the Cantor fan that is mixing, but not weakly continuum-wise fully expansive.*

Let $\sigma : C \longrightarrow C$ be the shift homeomorphism. Then it is well known the σ is mixing. Define the tent map $T : I \longrightarrow I$ by

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2] \\ 2 - 2x & \text{if } x \in [1/2, 1]. \end{cases}$$

It is well-known that the tent map is continuum-wise fully expansive and hence mixing. Let $g : X \longrightarrow X$ be defined by $g(x, c) = (T(x), \sigma(c))$. It follows from Proposition 9 that that G is mixing. However, $g([0, 1]_c) = [0, 1]_{\sigma(c)}$ for each c. Hence, g is not weakly continuum-wise fully expansive.

Example 20. *There exists a map on the Cantor fan that is weakly mixing, but not mixing.*

Let $T : [0, 1] \longrightarrow [0, 1]$ be the tent map and $\alpha : C \longrightarrow C$ be the map of the Cantor set stated in Section 1 that is weakly-mixing but not mixing. (See [6].) Then it follows from Propositions 10 and 11 that $h : X \longrightarrow X$ defined by $h(x, c) = (T(x), \alpha(c))$ is weakly mixing but not mixing.

5. Results on generalized $\sin(1/x)$-type continua

In this section, we give topological conditions for which two of the implications of Theorem 3 can be reversed, but a third cannot.

A map $f : X \longrightarrow y$ is *monotone* if $f^{-1}(y)$ is connected for each y in the range of f. An continuum X irreducible between a and b is a *λ-continuum* if there exists a monotone map $\phi : X \to [0, 1]$ such that $\phi(a) = 0$, $\phi(b) = 1$ and $\phi^{-1}(y)$ is a nowhere dense set in X for each $y \in [0, 1]$. The sets $\phi^{-1}(y)$ that are non-degenerate are called the *tranches* of X.

In the following, we will generalize the notion of a λ-continuum to allow monotone maps to arbitrary graphs instead of just the interval $[0, 1]$, but will need to further assume that there are not too many tranches. It is not difficult to construct a map of the so called "Cajun accordion" continuum (see [7]) – a λ-continuum for which $\phi^{-1}(y)$ is non-degenerate for every $y \in [0, 1]$ – which is mixing but not weakly continuum-wise fully expansive.

A continuum X is a *generalized* $\sin(1/x)$*-type continuum* if there exists a monotone map $\phi : X \to G$ onto a graph continuum G where

1. $\phi^{-1}(y)$ is a nowhere dense set in X for each $y \in G$
2. $\phi^{-1}(D)$ is dense in X where $D = \{y \in G \mid \phi^{-1}(y) \text{ is degenerate }\}$
3. if Y is a subcontinuum of $\phi^{-1}(y)$ and $\epsilon > 0$ then there exists an arc $[a, b] \subset G$ such that $\mathrm{d}_H(Y, \phi^{-1}([a, b])) < \epsilon$.

As above, the sets $\phi^{-1}(y)$ that are non-degenerate are called the *tranches* of X. Note that the standard $\sin(1/x)$ continuum is a generalized $\sin(1/x)$-type continuum. Note that 2) implies that D is dense in G.

For all of the results in this section, assume that $\phi : X \longrightarrow G$ is a monotone map of a generalized $\sin(1/x)$-continuum X onto a graph G with the above properties.

Proposition 21. *If X is a generalized* $\sin(1/x)$*-type continuum, then the set of subcontinua of X with nonempty interior is dense in the hyperspace $C(X)$.*

Proof. First, suppose that Y is a subcontinuum of X such that there exists an arc $(a,b) \subset G$ such that $(a,b) \subset \phi(Y)$. Then Y has nonempty interior. On the other hand, suppose that there exists $y \in [0,1]$ such that $Y \subset \phi^{-1}(y)$. Then, given $\epsilon > 0$, there an arc $[a,b] \subset G$ such that $d_H(Y, \phi^{-1}([a,b])) < \epsilon$. Since $\phi^{-1}([a,b])$ has nonempty interior, the result follows. □

Let $f : [-1,0) \longrightarrow \mathbb{R}^2$ be defined by $f(x) = (x, \sin(1/x) + 1/2)$ and $g : (0,1] \longrightarrow \mathbb{R}^2$ be defined by $f(x) = (x, \sin(1/x) - 1/2)$. Define $W = f([-1,0)) \cup (\{0\} \times [-3/2, 3/2]) \cup g((0,1])$. Then W is a λ-continuum. However, there is no subcontinuum of W with nonempty interior that is arbitrarily close to the subcontinuum $\{0\} \times [-1,1]$. Hence, W is not a generalized $\sin(1/x)$-type continuum.

Proposition 22. *Let U be a non-empty open set of X. Then there exists an open set V of G such that $\phi^{-1}(V) \subset U$.*

Proof. Recall that $D = \{y \in G \mid \phi^{-1}(y) \text{ is degenerate}\}$. Since $\phi^{-1}(D)$ is dense in G, there exists $x \in U$ such that $x = \phi^{-1}(\phi(x))$. Suppose on the contrary that there is no open set V that contains $\phi(x)$ such that $\phi^{-1}(V) \subset U$. Then there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in G such that $\lim_{n\to\infty} z_n = \phi(x)$ and $\phi^{-1}(z_n) \not\subset U$. Choose $x_n \in \phi^{-1}(z_n) - U$ and without loss of generality assume $\widehat{x} = \lim_{n\to\infty} x_n$. Notice that $x \neq \widehat{x}$. Then

$$\phi(\widehat{x}) = \lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} z_n = \phi(x).$$

However, this contradicts the fact that $x = \phi^{-1}(\phi(x))$. □

The following corollary follows from the fact that every non-empty open set of a graph contains an open interval.

Corollary 23. *Let U be a non-empty open set of X. Then there exist an open interval (a,b) of G such that $\phi^{-1}((a,b)) \subset U$.*

Proposition 24. *Suppose that $f : G \longrightarrow G$ is weakly mixing and $F : X \longrightarrow X$ is such that $f \circ \phi = \phi \circ F$. Then F is weakly mixing.*

Proof. Suppose that U_1, U_2, V_1, V_2 are non-empty open sets of X. Then there exist open intervals A_1, A_2, B_1, B_2 of G such that $\phi^{-1}(A_1) \subset U_1$, $\phi^{-1}(A_2) \subset U_2$, $\phi^{-1}(B_1) \subset V_1$ and $\phi^{-1}(B_2) \subset V_2$ by Corollary 23. Since f is weakly mixing, there exists $m \geq 0$ such that $f^m(A_1) \cap B_1 \neq \emptyset$ and $f^m(A_2) \cap B_2 \neq \emptyset$. Hence, $F^m(U_1) \cap V_1 \supset \phi^{-1}(f^m(A_1)) \cap \phi^{-1}(B_1) \neq \emptyset$ and $F^m(U_2) \cap V_2 \supset \phi^{-1}(f^m(A_2)) \cap \phi^{-1}(B_2) \neq \emptyset$. □

Theorem 25. *Suppose that $f : X \longrightarrow X$ is a weakly mixing map of a generalized $\sin(1/x)$-type continuum, then f is weakly continuum-wise fully expansive.*

Proof. Let (a,b) be an open interval in G and choose $a < x_1 < y_1 < x_2 < y_2 < b$. Replace $(a, x_1), (x_1, y_1), (x_2, y_2)$ and (y_1, x_2) with $\phi^{-1}((a, x_1)), \phi^{-1}((x_1, y_1)), \phi^{-1}((x_2, y_2))$ and $\phi^{-1}((y_1, x_2))$ respectively in the proof of Lemma 14. It follows that there is an $n \geq 0$ such that $\phi^{-1}((x_1, y_1)) \subset f^n(\phi^{-1}((x_1, y_1)))$ and $\phi^{-1}((x_2, y_2)) \subset f^n(\phi^{-1}((x_2, y_2)))$. Let V be any non-empty open set in X. Then there exists an open interval $(c,d) \subset G$ such that $\phi^{-1}((c,d)) \subset V$. Since f is weakly mixing, there exists an m such that $f^m(\phi^{-1}((c,d))) \cap \phi^{-1}((a, x_1)) \neq \emptyset$ and $f^m(\phi^{-1}((c,d))) \cap \phi^{-1}((y_1, x_2)) \neq \emptyset$. Thus either $\phi^{-1}((x_1, y_1)) \subset f^m(\phi^{-1}((c,d))) \subset f^m(V)$ or $\phi^{-1}((x_2, y_2)) \subset f^m(\phi^{-1}((c,d))) \subset f^m(V)$. It now follows from Lemma 8 that f is mixing. By Proposition 21, the subcontinua of X with nonempty interior are dense in $C(X)$. Hence, it follows that f is weakly continuum-wise fully expansive. □

Example 26. *There exists a map of a generalized* $\sin(1/x)$*-type continuum that is weakly continuum-wise fully expansive but not continuum-wise fully expansive (or even continuum-wise expansive).*

If $Y \subset \mathbb{R}^2$, let $\pi_1 : Y \longrightarrow \mathbb{R}$ be the projection map of the first coordinate of Y. If $Y_1, Y_2 \subset \mathbb{R}^2$ such that $\pi_1(Y_1) = \pi_1(Y_2)$, define $\mathrm{D}_2(Y_1, Y_2) = \sup\{|y_1 - y_2| \mid (x, y_1) \in Y_1$ and $(x, y_2) \in Y_2\}$. Let $X_1 = T_0 \cup (-\infty, \infty)_1 \cup T_1 \subset \mathbb{R}^2$ defined in the following way:

(1) $T_0 = \{(0, y) \mid y \in [0, 1]\}$
(2) $T_1 = \{(1, y) \mid y \in [0, 1]\}$
(3) $(-\infty, \infty)_1$ is a copy of $\mathbb{R}$ such that
  (a) $(-\infty, 0]_1$ limits to T_0 such that $(-\infty, 0]_1 \cup T_0$ is homeomorphic to the standard $\sin(1/x)$ continuum
  (b) $[0, \infty)_1$ limits to T_1 such that $T_1 \cup [0, \infty)_1$ is homeomorphic to the standard $\sin(1/x)$ continuum
  (c) π_1 maps $(-\infty, \infty)_1$ homeomorphically onto $(0, 1)$.

Let $\phi_1 : X_1 \longrightarrow [0, 1]$ be defined by $\phi_1(x) = \pi_1(x)$. Then X_1 is a generalized $\sin(1/x)$-type continuum with tranches $T_0 = \phi_1(0)$ and $T_1 = \phi_1(1)$.

Let δ_1 be chosen such that $\mathrm{diam}(\phi_1^{-1}((1/2 - \delta_1, 1/2 + \delta_1))) < 1/2$. Let $T_{1/2}$ be a vertical line segment with diameter $1/2$ and midpoint at $\phi_1^{-1}(1/2)$. Let $[1, \infty)_{1/2}$ and $(-\infty, -1]_{1/2}$ be rays in $\mathbb{R}^2$ that limit to $T_{1/2}$ such that

(1) the endpoint $1_{1/2}$ of $[1, \infty)_{1/2}$ is equal to $\phi_1^{-1}(1/2 - \delta_1)$
(2) the endpoint $-1_{1/2}$ of $(-\infty, -1]_{1/2}$ is equal to $\phi_1^{-1}(1/2 + \delta_1)$
(3) π_1 maps $[1, \infty)_{1/2}$ homeomorphically onto $[1/2 - \delta_1, 1/2)$
(4) π_1 maps $(-\infty, -1]_{1/2}$ homeomorphically onto $(1/2, 1/2 + \delta_1, 1/2]$
(5) $\mathrm{D}_2([1, \infty)_{1/2} \cup (-\infty, -1]_{1/2}, \phi_1^{-1}([1/2 - \delta_1) \cup (1/2, 1/2 + \delta_1])) < 1/2$.

Let $X_2 = (X_1 - \phi_1^{-1}((1/2 - \delta_1, 1/2 + \delta_)) \bigcup([1, \infty)_{1/2} \cup T_{1/2} \cup (-\infty, -1]_{1/2}$ and $\phi_2 : X_2 \longrightarrow [0, 1]$ be defined by $\phi_2(x) = \pi_1(x)$. Let $Q_2 = \{0, 1/2, 1\}$. Then X_2 is a generalized $\sin(1/x)$-type continuum and ϕ_{n+1} is a monotone map with tranches $\{\phi_2^{-1}(q)\}_{q \in Q_2}$.

Continuing inductively, suppose that X_n and ϕ_n have been defined. Then there exists $0 < \delta_n < 1/2^n$ such that $\mathrm{diam}(\phi_n^{-1}(((2k+1)/2^n - \delta_n, (2k+1)/2^n + \delta_n)) < 1/2^n$ for each $k \in \{0, ..., 2^{n-1} - 1\}$. Let $T_{(2k+1)/2^n}$ be a vertical line segment with diameter $1/2^n$ and midpoint at $\phi_1^{-1}((2k+1)/2^n)$. Let $[1, \infty)_{(2k+1)/2^n}$ and $(-\infty, -1]_{(2k+1)/2^n}$ be rays in $\mathbb{R}^2$ that limit to $T_{(2k+1)/2^n}$ such that

(1) the endpoint $1_{(2k+1)/2^n}$ of $[1, \infty)_{(2k+1)/2^n}$ is equal to $\phi_n^{-1}((2k+1)/2^n - \delta_n)$
(2) the endpoint $-1_{(2k+1)/2^n}$ of $(-\infty, -1]_{(2k+1)/2^n}$ is equal to $\phi_n^{-1}((2k+1)/2^n + \delta_n)$
(3) π_1 maps $[1, \infty)_{(2k+1)/2^n}$ homeomorphically onto $[(2k+1)/2^n - \delta_n, (2k+1)/2^n)$
(4) π_1 maps $(-\infty, -1]_{(2k+1)/2^n}$ homeomorphically onto $((2k+1)/2^n, (2k+1)/2^n + \delta_n, (2k+1)/2^n]$
(5) $\mathrm{D}_2([1, \infty)_{(2k+1)/2^n} \cup (-\infty, -1]_{(2k+1)/2^n}, \phi_n^{-1}([(2k+1)/2^n - \delta_n) \cup ((2k+1)/2^n, (2k+1)/2^n + \delta_n])) < 1/2^n$.

Let $X_{n+1} = (X_n - \bigcup_{k=0}^{2^{n-1}-1} \phi_n^{-1}(((2k+1)/2^n - \delta_n, (2k+1)/2^n + \delta_n)) \bigcup_{k=0}^{2^{n-1}-1}([1, \infty)_{(2k+1)/2^n} \cup T_{(2k+1)/2^n} \cup (-\infty, -1]_{(2k+1)/2^n}$ and $\phi_{n+1} : X_{n+1} \longrightarrow [0, 1]$ be defined by $\phi_{n+1}(x) = \pi_1(x)$. Let $Q_{n+1} = Q_n \cup \{(2k+1)/2^n \mid k \in \{0, ..., 2^{n-1} - 1\}\}$. Then X_{n+1} is a

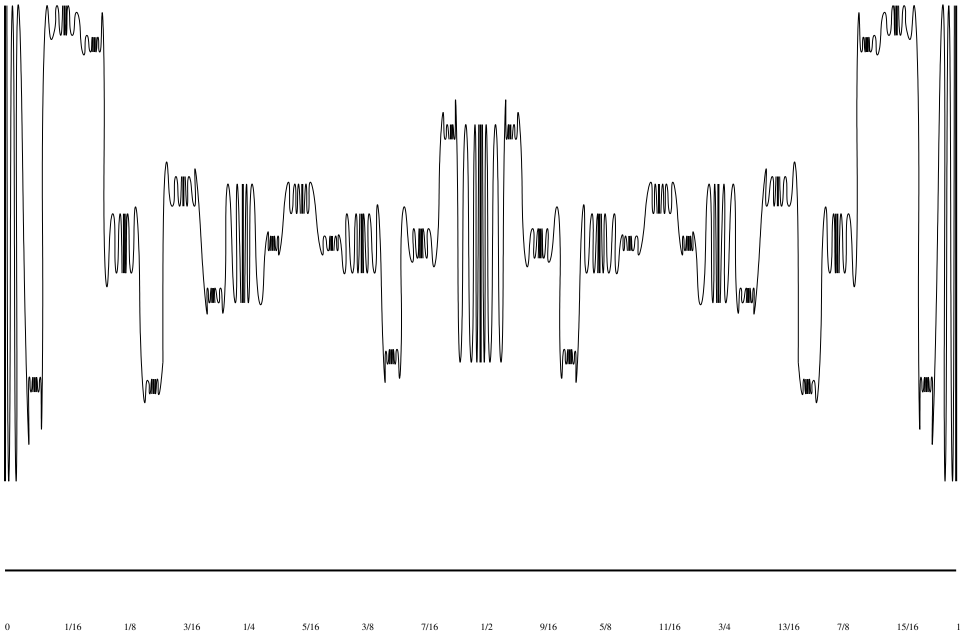


FIGURE 3. The generalized $\sin(1/x)$-type continuum X.

generalized $\sin(1/x)$-type continuum and ϕ_{n+1} is a monotone map with tranches $\{\phi_{n+1}^{-1}(q)\}_{q \in Q_{n+1}}$.

Let $X = \bigcap_{n\to\infty} \overline{\bigcup_{k\geq n} X_k}$. See Figure 3 for an illustration. Notice that for every $m \geq n+1$, $Y^k_{1/2^n} \subset X_m$. Let $Q = \bigcup_{n=1}^{\infty} Q_n$. If $y \in [0,1] - Q$, then $\mathrm{d}(\phi_{n+1}^{-1}(y), \phi_n^{-1}(y)) < 1/2^n$. Hence $\{\phi_n^{-1}(y)\}_{n=1}^{\infty}$ converges to a single point in X, say $x(y)$. Hence we can define the monotone map $\phi : X :\longrightarrow [0,1]$ by $\phi(x) = (2k+1)/2^n$ if $x \in T_{(2k+1)/2^n}$ and $\phi(x) = y$ if $x = x(y)$.

Now let $\widehat{T} : [0,1] \longrightarrow [0,1]$ be the tent map and notice that $\widehat{T}(Q) = Q$ and $\widehat{T}([0,1] - Q) = [0,1] - Q$. Define $F : X \longrightarrow X$ such that F maps T_q linearly onto $T_{f(q)}$ and $F(x) = \phi^{-1}(\widehat{T}(\phi(x)))$ if $\phi(x) \in [0,1] - Q$. Then $F \circ \phi = \phi \circ F$. Since $F(T_0) = T_0$, F is not continuum-wise fully expansive. However, since $\widehat{T}$ is continuum-wise fully expansive and hence weakly mixing, it follows from Proposition 24 and Theorem 25 that F is weakly mixing and hence weakly continuum-wise fully expansive.

Notice that by replacing each $T_{(2k+1)/2^n}$ with an appropriate copy of W described earlier above Proposition 22, we can construct a continuum $\widehat{X}$ similar but not homeomorphic to X ($\widehat{X}$ is not a generalized $\sin(1/x)$-type continuum) and a similar function $\widehat{F} : \widehat{X} \longrightarrow \widehat{X}$ that is semi-conjugate to the tent map. However, this map is mixing but not weakly continuum-wise fully expansive.

Now the following diagram is complete:

	weakly mixing ⇒ mixing	mixing ⇒ weakly c.w.f.e.	weakly c.w.f.e. ⇒ c.w.f.e.
graph	Yes	Yes	Yes
gen. $\sin(1/x)$-type	Yes	Yes	No
dendrite	No	Yes	No
Cantor fan	No	No	No

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